

Symmetric Potentials of Gauged Supergravities in Diverse Dimensions and Coulomb Branch of Gauge Theories

M. Cvetič^{†§1}, S.S. Gubser^{*2}, H. Lü^{†1} and C.N. Pope^{‡*3}

[†]*Dept. of Phys. and Astro., University of Pennsylvania, Philadelphia, PA 19104*

[‡]*Center for Theoretical Physics, Texas A&M University, College Station, TX 77843*

^{*}*Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138*

[§]*Institute for Theor. Phys., Univ. of Santa Barbara, Santa Barbara, CA 93106*

^{*}*SISSA, Via Beirut, No. 2-4, 34013 Trieste, Italy*

ABSTRACT

A class of conformally flat and asymptotically anti-de Sitter geometries involving profiles of scalar fields is studied from the point of view of gauged supergravity. The scalars involved in the solutions parameterise the $SL(N, \mathbb{R})/SO(N)$ submanifold of the full scalar coset of the gauged supergravity, and are described by a symmetric potential with a universal form. These geometries descend via consistent truncation from distributions of D3-branes, M2-branes, or M5-branes in ten or eleven dimensions. We exhibit analogous solutions asymptotic to AdS_6 which descend from the D4-D8-brane system. We obtain the related six-dimensional theory by consistent reduction from massive type IIA supergravity. All our geometries correspond to states in the Coulomb branch of the dual conformal field theories. We analyze linear fluctuations of minimally coupled scalars and find both discrete and continuous spectra, but always bounded below.

Contents

1	Introduction	2
2	Symmetric Potentials in Gauged Supergravities, and Domain-wall Solutions	3
2.1	Symmetric potentials	4
2.2	AdS domain-wall solutions	6
3	Higher-dimensional Origin	9
4	Domain Walls as Distributed p-branes	13
4.1	AdS ₇ domain walls as distributed M5-branes	13
4.2	AdS ₄ domain walls as distributed M2-branes	15
4.3	AdS ₅ domain walls as distributed D3-branes	16
5	Six-dimensional Gauged Supergravity	18
5.1	$D = 6$ potential	20
5.2	AdS ₆ domain walls as distributed D4-D8-branes	21
6	Analysis of the spectrum	23
6.1	Spectrum with “Euclidean” parameters	24
6.2	Spectrum with “Lorentzian” parameters	27

1 Introduction

The AdS/CFT correspondence [1, 2, 3] offers the possibility of extracting information about gauge theory in a strongly coupled limit from a dual supergravity description. Much recent work (see for instance [4, 5, 6, 7, 8]) has been devoted to bulk geometries that are only asymptotically anti-de Sitter and retain Poincaré invariance in the boundary directions. Such geometries correspond either to a relevant deformation of the conformal field theory or a state of the conformal field theory with a VEV which sets an energy scale. The geometries arise from solving a system of scalars coupled to gravity. At the linearized level, the second order scalar equations admit two linearly independent solutions. Modulo the subtleties discussed in [9], a bulk geometry corresponds to a deformation of the conformal field theory if the scalar profiles approach the more singular of the two possible linearized solutions near the boundary, and it corresponds to a state of the conformal field theory if they approach the less singular solutions.

We shall be concerned in this paper with fairly broad class of asymptotically anti-de Sitter solutions which can nevertheless be obtained in closed form. These geometries lift to solutions of ten or eleven-dimensional supergravity describing some distribution of parallel D3-branes, M2-branes, or M5-branes. They all preserve one half of supersymmetry, and they are states of the dual conformal field theories: the VEV's in question are simply the positions of the branes in the distribution.

These geometries are interesting for several reasons. First, a subset of them arise as limits of charged black holes in anti-de Sitter space. The limit in question leads (generically) to a naked timelike singularity. When the asymptotically anti-de Sitter solution is lifted to ten or eleven dimensions, the only singularities are due to the fact that the branes are distributed continuously. They are naked null singularities similar to those around low-dimensional D-branes.

Second, the geometries provide examples of consistent truncation in various dimensions. The explicit embedding of lower dimensional gauged supergravities in higher dimensions has been worked out only for a handful of cases, namely $N = 2$ (maximal) and $N = 1$ gauged $D = 7$ supergravity as S^4 reductions from $D = 11$ [10, 11], and $D = 6$ gauged $N = 2$ supergravity as a warped S^4 reduction from massive type IIA supergravity [12]. (For the S^7 reduction of $D = 11$ supergravity, the results of [13, 14] are complete in principle but very implicit in form.) Consistent embeddings in $D = 11$ and $D = 10$ type IIB for the gauged supergravities with the maximal abelian subgroups $U(1)^4$, $U(1)^2$ and $U(1)^3$ of the $SO(8)$, $SO(5)$ and $SO(6)$ gauge groups from the maximal S^7 , S^4 and S^5 reductions have

also been constructed, in [15]. It is therefore of interest in the ongoing study of consistent truncations to have further examples of some generality.

Some of the geometries we consider appeared in [16, 17], and a set of representative examples was studied in [7]. In [7] the curious result was obtained that some of the asymptotically anti-de Sitter geometries lifted to distributions of branes of indefinite sign: that is, there were “branes” of negative charge and negative tension present in the distribution. Negative charge and positive tension would indicate an anti-brane. But negative charge and negative tension, in the absence of some orientifolding, is pathological. For instance, the transverse fluctuations of such branes would have negative kinetic terms. Yet the asymptotically anti-de Sitter geometries that lead to these disallowed brane distributions are no more pathological than those where the brane distribution is strictly positive.

The paper is organized as follows. In section 2, we describe the interacting scalars which participate in the conformally flat and asymptotically anti-de Sitter geometries. Because of the similarities of these geometries with supergravity domain wall solutions (see for example [19]), we will often refer to them as anti-de Sitter domain walls. In section 3 we explain the ten and eleven dimensional origin of the geometries. In section 4 we describe explicitly the brane distributions corresponding to each type of domain wall. In section 5, we consider asymptotically AdS_6 geometries which arise from many D4-branes distributed inside coincident D8-branes. We will see that D3-branes, M2-branes, and M5-branes behave quite similarly in their respective dimensions, and the geometries and distributions can be described very much in parallel. The D4-D8-brane solutions are rather different and provide another interesting example of consistent truncation. Finally, in section 6, we analyze the spectrum of minimally coupled scalars in the various domain wall geometries we have constructed.

2 Symmetric Potentials in Gauged Supergravities, and Domain-wall Solutions

In this section, we consider the structure of the scalar potentials that arise in the maximally-supersymmetric gauged supergravities in $D = 4, 5$ and 7 dimensions. We shall present the results in a slightly more general framework for arbitrary dimension D , although in the end we shall see that $D = 4, 5$ and 7 are singled out.

2.1 Symmetric potentials

The scalar fields in D -dimensional maximal supergravity parameterize the coset E_{11-D}/K , where E_n is the maximally-non-compact form of the exceptional group E_n , and K is its maximal compact subgroup.¹ We can focus on the $SL(N, \mathbb{R})$ subgroup of E_n , and consider in particular the subset of $\frac{1}{2}N(N+1)-1$ scalars parameterizing the coset $SL(N, \mathbb{R})/SO(N)$. (In $D = 4$ and $D = 5$, the relevant subgroups of E_7 and E_6 are $SL(8, \mathbb{R})$ and $SL(6, \mathbb{R})$. In $D = 7$, the group E_4 is itself just A_4 , and so we consider $SL(5, \mathbb{R})$ in that case.)

One can use the local $SO(N)$ transformations in order to diagonalize the scalar potential. Thus we are led to consider the following Lagrangian for gravity plus scalars in D dimensions:

$$e^{-1} \mathcal{L}_D = R - \frac{1}{2}(\partial\vec{\varphi})^2 - V, \quad (2.1)$$

where the potential V is given by

$$V = -\frac{1}{2}g^2 \left(\left(\sum_{i=1}^N X_i \right)^2 - 2 \sum_{i=1}^N X_i^2 \right). \quad (2.2)$$

(In $D = 4, 5$ and 7 , we shall have $N = 8, 6$ and 5 respectively.) The N quantities X_i , which are subject to the constraint

$$\prod_{i=1}^N X_i = 1, \quad (2.3)$$

can be parameterized in terms of $(N-1)$ independent dilatonic scalars $\vec{\varphi}$ as follows:

$$X_i = e^{-\frac{1}{2}\vec{b}_i \cdot \vec{\varphi}}, \quad (2.4)$$

where the \vec{b}_i are the weight vectors of the fundamental representation of $SL(N, \mathbb{R})$, satisfying

$$\vec{b}_i \cdot \vec{b}_j = 8\delta_{ij} - \frac{8}{N}, \quad \sum_i \vec{b}_i = 0, \quad (\vec{u} \cdot \vec{b}_i) \vec{b}_i = 8\vec{u}, \quad (2.5)$$

where \vec{u} is an arbitrary N -vector. The last equation in (2.5) allows us to express the dilatons $\vec{\varphi}$ in terms of the X_i :

$$\vec{\varphi} = -\frac{1}{4} \sum_i \vec{b}_i \log X_i. \quad (2.6)$$

Note that the potential has a minimum at $X_i = 1$ for $N \leq 3$, a point of inflection at $X_i = 1$ for $N = 4$, and a maximum at $X_i = 1$ for $N \geq 5$.

The equations of motion for the scalar fields, following from (2.1), are

$$\square \vec{\varphi} = \frac{\partial V}{\partial \vec{\varphi}}. \quad (2.7)$$

¹As usual, we define $E_5 \sim D_5$, $E_4 \sim A_4$, $E_3 \sim A_2 \times A_1$, $E_2 \sim A_1 \times \mathbb{R}$ and $E_1 \sim \mathbb{R}$.

From (2.4) it follows that $\partial X_i / \partial \vec{\varphi} = -\frac{1}{2} \vec{b}_i X_i$, and hence the equations of motion (2.7) become

$$\square \vec{\varphi} = \frac{1}{2} g^2 \sum_i \vec{b}_i \left(X_i \sum_j X_j - 2X_i^2 \right). \quad (2.8)$$

Note that we can also write the scalar equations of motion as

$$\square \log X_i = 2g^2 \left(2X_i^2 - X_i \sum_j X_j - \frac{2}{N} \sum_j X_j^2 + \frac{1}{N} \left(\sum_j X_j \right)^2 \right). \quad (2.9)$$

(The last two terms on the right-hand side imply that the sum over i is zero, as it must be since the N quantities X_i satisfy (2.3).)

In order to make contact with previous results, we can verify that if the potential (2.2) is specialized to the case where as many pairs of the quantities X_i as possible are set equal, we recover the previously-known potentials that arise when the various gauged supergravities are truncated to the multiplets comprising the maximal abelian subgroups of the original gauge groups. Thus for $D = 4$ we may set the 8 quantities X_i pairwise equal,

$$X_1 = X_2 \equiv \tilde{X}_1, \quad X_3 = X_4 \equiv \tilde{X}_2, \quad X_5 = X_6 \equiv \tilde{X}_3, \quad X_7 = X_8 \equiv \tilde{X}_4, \quad (2.10)$$

whereupon the potential becomes

$$V = -4g^2 \sum_{a < b} \tilde{X}_a \tilde{X}_b, \quad (2.11)$$

where the four \tilde{X}_a satisfy $\prod_a \tilde{X}_a = 1$. This potential is the one encountered in [18, 15], for the scalars in the truncation of four-dimensional gauged $SO(8)$ supergravity to its $U(1)^4$ maximal abelian subgroup.

For $D = 5$, we may set the 6 quantities X_i pairwise equal,

$$X_1 = X_2 \equiv \tilde{X}_1, \quad X_3 = X_4 \equiv \tilde{X}_2, \quad X_5 = X_6 \equiv \tilde{X}_3, \quad (2.12)$$

under which the potential V given in (2.2) reduces to

$$V = -4g^2 \sum_{a=1}^3 \tilde{X}_a^{-1}, \quad (2.13)$$

where the three scalar quantities \tilde{X}_a satisfy $\prod \tilde{X}_a = 1$. This potential was encountered in [20, 15]; it arises in the truncation of the five-dimensional gauged $SO(6)$ supergravity to its $U(1)^3$ maximal abelian subgroup.

Finally, in $D = 7$ we may set

$$X_1 = X_2 \equiv \tilde{X}_1, \quad X_3 = X_4 \equiv \tilde{X}_2, \quad X_5 \equiv \tilde{X}_0. \quad (2.14)$$

The three \tilde{X}_a satisfy $\tilde{X}_0 = (\tilde{X}_1 \tilde{X}_2)^{-2}$, and the potential (2.2) becomes

$$V = -\frac{1}{2}g^2(-\tilde{X}_0^2 + 8\tilde{X}_1 \tilde{X}_2 + 4\tilde{X}_0 \tilde{X}_1 + 4\tilde{X}_0 \tilde{X}_2). \quad (2.15)$$

This can be recognized as precisely the potential previously encountered in the truncation of seven-dimensional $SO(5)$ gauged supergravity to its $U(1)^2$ maximal abelian subgroup [15]. It is interesting to observe how the “asymmetry” between the \tilde{X}_0 field and the \tilde{X}_1 and \tilde{X}_2 fields here can be simply understood as originating from the fact that the more general potential (2.2) has an odd number of fields X_i in this case, and so there is inevitably an “odd one out” after setting as many pairs as possible equal.

It should be emphasized that in each case, setting pairs of X_i equal is consistent with the equations of motion (2.9).

2.2 AdS domain-wall solutions

The equations of motion following from the gravity plus scalar Lagrangian (2.1) comprise the scalar equations (2.8), together with the Einstein equation

$$R_{MN} = \frac{1}{2}\partial_M\vec{\varphi} \cdot \partial_N\vec{\varphi} + \frac{1}{D-2}V g_{MN}. \quad (2.16)$$

We find that these equations admit solutions given by

$$\begin{aligned} ds_D^2 &= (g r)^{\frac{4}{D-3}} \left(\prod_i H_i \right)^{\frac{1}{2} - \frac{2}{N}} dx^\mu dx_\mu + \left(\prod_i H_i \right)^{-\frac{2}{N}} \frac{dr^2}{g^2 r^2}, \\ X_i &= H_i^{-1} \left(\prod_j H_j \right)^{\frac{1}{N}}, \end{aligned} \quad (2.17)$$

where

$$H_i = 1 + \frac{\ell_i^2}{r^2}, \quad (2.18)$$

provided that the integer N is related to the spacetime dimension D by

$$N = \frac{4(D-2)}{D-3}. \quad (2.19)$$

Note that for $D = 4, 5$ and 7 this indeed implies $N = 8, 6$ and 5 respectively. As we explained in the Introduction, we shall refer to solutions of this kind as anti-de Sitter domain walls.

In verifying these solutions, it is useful first to calculate the Ricci tensor for the class of metrics

$$ds_D^2 = e^{2A} dx^\mu dx_\mu + e^{2B} dr^2, \quad (2.20)$$

where A and B are functions of r . In the orthonormal basis $e^\mu = e^A dx^\mu$, $e^r = e^B dr$, we find that the tangent-space components of the Ricci tensor are

$$\begin{aligned} R_{\mu\nu} &= -(A'' + (D-1)A'^2 - A'B')e^{-2B}\eta_{\mu\nu}, \\ R_{rr} &= -(D-1)(A'' + A'^2 - A'B')e^{-2B}, \end{aligned} \quad (2.21)$$

where a prime denotes a derivative with respect to r . Thus the Einstein equation (2.16) becomes

$$\begin{aligned} -(A'' + (D-1)A'^2 - A'B')e^{-2B} &= \frac{1}{D-2}V, \\ -(D-1)(A'' + A'^2 - A'B')e^{-2B} &= \frac{1}{2}e^{-2B}\vec{\varphi}' \cdot \vec{\varphi}' + \frac{1}{D-2}V. \end{aligned} \quad (2.22)$$

It is useful also to note that (2.18) implies $H'_i = (-2/r)(H_i - 1)$. Substituting into the Einstein and scalar equations of motion, we now straightforwardly verify that (2.17) is a solution.

It should be noted that the solutions we have obtained here reduce to previously-known ones in certain special cases. Namely, if we make the pairwise identifications $\ell_{2p-1} = \ell_{2p} \equiv L_p$ for $1 \leq p \leq N/2$ in $D = 4$ or $D = 5$, then we obtain solutions with 4 or 3 parameters L_p respectively, which correspond precisely to the extremal BPS limits of the 4-charge [18, 15] and 3-charge AdS_4 [20] and AdS_5 black holes of maximal $D = 4$ and $D = 5$ supergravity. In the $D = 7$ case, if we set $\ell_1 = \ell_2 \equiv L_1$, $\ell_3 = \ell_4 \equiv L_2$, and $\ell_5 = 0$, we recover the previously-known 2-parameter solutions that arise as the extremal BPS limits of the 2-charge AdS_7 black holes [15] of maximal gauged $D = 7$ supergravity.²

It should be emphasized also that although we naively appear to have N parameters ℓ_i in the general solutions (2.17), there are really only $(N-1)$ genuinely independent ones. To see this, let us suppose, without loss of generality, that ℓ_N^2 is the smallest of all the parameters, *i.e.* $\ell_i^2 \geq \ell_N^2$. We also decompose the index i into $i = (a, N)$, where $1 \leq a \leq N-1$. We now make the following coordinate transformation,

$$r^2 = R^2 - \ell_N^2, \quad (2.23)$$

and at the same time define

$$\tilde{H}_a = 1 + \frac{\lambda_a^2}{R^2}, \quad \lambda_a^2 \equiv \ell_a^2 - \ell_N^2. \quad (2.24)$$

Straightforward algebra now shows that in terms of these quantities, the metric in the solution (2.17) becomes

$$ds_D^2 = (gR)^{\frac{4}{D-3}} \left(\prod_a \tilde{H}_a \right)^{\frac{1}{2} - \frac{2}{N}} dx^\mu dx_\mu + \left(\prod_a \tilde{H}_a \right)^{-\frac{2}{N}} \frac{dR^2}{g^2 R^2}, \quad (2.25)$$

²In all cases the charges vanish when the extremal limit is taken.

where the products are now only over the $(N - 1)$ functions \tilde{H}_a . The quantities X_i in (2.17) become

$$X_a = \tilde{H}_a^{-1} \left(\prod_b \tilde{H}_b \right)^{\frac{1}{N}}, \quad X_N = \left(\prod_b \tilde{H}_b \right)^{\frac{1}{N}}. \quad (2.26)$$

Thus we see that after making the coordinate transformation (2.23), the solution (2.17) with N parameters ℓ_i^2 is nothing but a solution of the identical form, but with transformed parameters

$$\ell_i^2 = (\ell_1^2, \ell_2^2, \dots, \ell_{N-1}^2, \ell_N^2) \longrightarrow (\ell_1^2 - \ell_N^2, \ell_2^2 - \ell_N^2, \dots, \ell_{N-1}^2 - \ell_N^2, 0). \quad (2.27)$$

Thus there are actually only $(N - 1)$ independent parameters in the solutions (2.17), implying 7, 6 and 4 in $D = 4, 5$ and 7 respectively.

A simple example of the transformation (2.27) is the case where all the ℓ_i have a common value, ℓ . In this case the above transformation leads to a solution where all the ℓ_i vanish, and the geometry is locally exactly anti-de Sitter. It is possible however for the geometry to change abruptly at some radius. This is the case, for example, if one has a spherical shell of D3-branes at a radius ℓ . Outside the shell, the geometry is $\text{AdS}_5 \times S^5$, exactly the same as if the shell were collapsed to a point. Inside the shell the geometry is ten-dimensional flat space. This configuration has been considered in [16, 22, 23]. It was pointed out already in [16] and emphasized in [23] that the background outside the shell is insensitive to what spherical distribution of branes one puts inside the shell: only the total number of branes controls the overall curvature.

The transformation (2.27) is a generalized shell theorem: as we will see in later sections, it effectively exhibits a class of brane distributions that all have the same exterior.

Further applications of the transformation (2.27) arise if we consider a situation where m of the ℓ_i^2 parameters are equal. Suppose we order the parameters so that $\ell_{N-m+1}^2 = \ell_{N-m+2}^2 = \dots = \ell_N^2 \equiv L^2$. Then, after making the transformation (2.27), we will have new parameters

$$\tilde{\ell}_i^2 = (\tilde{\ell}_1^2, \tilde{\ell}_2^2, \dots, \tilde{\ell}_{N-m}^2, 0, 0, \dots, 0), \quad (2.28)$$

where $\tilde{\ell}_i^2 = \ell_i^2 - L^2$, and so the configuration is mapped to one with only $(N - m)$ non-vanishing parameters. If L^2 is the smallest value among the original parameters, then all the transformed parameters $\tilde{\ell}_i^2$ will be positive. However, if some of the original parameters were smaller than L^2 , then after the transformation they will correspond to $\tilde{\ell}_i^2$ parameters that are negative.

In fact, quite generally we can use the transformation to map the sets of solutions with purely non-negative ℓ_i^2 parameters into sets of solutions with arbitrary numbers p and q

of positive and negative ℓ_i^2 , with $0 \leq p + q \leq N$. For example, the solution with $\ell_2^2 = \ell_3^2 = \dots = \ell_N^2 \equiv L^2 > 0$ and $\ell_1^2 = 0$ transforms into a solution with $\tilde{\ell}_i^2 = (-L^2, 0, 0, \dots, 0)$, with $\tilde{\ell}_1^2 < 0$. Thus a special case of the $n = N - 1$ solutions with positive ℓ_i^2 transforms to an $n = 1$ solution with negative ℓ_1^2 . Likewise, we can obtain a solution with n negative ℓ_i^2 parameters from a solution with $(N - n)$ positive parameters. These transformations will be useful later, when we derive results for the charge distributions that allow us to view the domain-wall solutions oxidized to higher dimensions as distributions of M-branes or D-branes. Thus having obtained results for the distributions for arbitrary numbers n of positive ℓ_i^2 parameters, we can immediately read off results for sets of negative parameters. This mapping between solutions with positive ℓ_i^2 parameters and negative parameters will also be useful when we consider the spectra of wave equations in these background geometries in section 6. In general, for reasons explained in section 6.1, we refer to parameters with $\ell_i^2 > 0$ as “Lorentzian,” and parameters with $\ell_i^2 < 0$ as “Euclidean.”

3 Higher-dimensional Origin

In the previous section, we considered the D -dimensional theories of gravity plus scalar fields that arise as consistent truncations of the gauged maximal supergravities in $D = 4, 5$ and 7 , and we showed that they admit multi-parameter domain-wall solutions. It is believed that the gauged maximal supergravities can all be obtained as consistent Kaluza-Klein reductions using sphere compactifications of certain higher-dimensional supergravities. Specifically, it has been shown that the $D = 4$ and $D = 7$ gauged theories arise from the S^7 and S^4 reductions of eleven-dimensional supergravity, while the $D = 5$ gauged theory is expected to arise from the S^5 reduction of type IIB supergravity.³ We can therefore expect that the truncations of the various gauged supergravities that we considered in the previous section should themselves be directly derivable as consistent Kaluza-Klein reductions from the relevant higher-dimensional supergravities.

In this section, we shall show explicitly how the various D -dimensional theories that we discussed in section 2 can be embedded into the higher-dimensional supergravities. The crucial point will be that these are *consistent* embeddings, meaning that all solutions of the lower-dimensional theories will be solutions of the higher-dimensional ones. In each case, the scalar fields of the lower-dimensional theory will appear as parameters describing inho-

³It has also been shown that the largest gauged supergravity in $D = 6$ arises as an S^4 reduction of massive type IIA supergravity [12].

ogeneous deformations of the compactifying sphere metric. Because of the inhomogeneity, demonstrating the *consistency* of the reduction procedure is highly non-trivial, since there is no simple group-invariance argument that can account for it. In fact a crucial point in all such Kaluza-Klein reductions is that the scalar fields appear not only in the metric reduction Ansatz, but also in the reduction Ansatz for an antisymmetric tensor field of the higher-dimensional supergravity. It is only because of remarkable “conspiracies” between the contributions from the scalars in the metric and field strength Ansätze that the consistent reduction is possible. Thus the issue of consistency is one that can be addressed only if the complete reduction Ansätze, including those for the antisymmetric tensor fields, are given.

Let us first present our results for the consistent reduction Ansätze. We can give all three cases, for the reductions to $D = 4, 5$ and 7 , in a single set of formulae. In each case the relevant part of the higher-dimensional supergravity theory comprises the metric $d\hat{s}^2$, and an antisymmetric tensor field strength $\hat{F}_{(D)}$.⁴ For the case of $D = 4$, we have the 4-form field $\hat{F}_{(4)}$ of eleven-dimensional supergravity. For $D = 7$, we have $\hat{F}_{(7)} = \hat{*}\hat{F}_{(4)}$, where $\hat{F}_{(4)}$ is again the 4-form of eleven-dimensional supergravity. For $D = 5$, we have $\hat{F}_{(5)}$, from which the self-dual 5-form $\hat{H}_{(5)}$ will be constructed as $\hat{H}_{(5)} = \hat{F}_{(5)} + \hat{*}\hat{F}_{(5)}$. Our expressions for the metric and field-strength Ansätze are

$$\begin{aligned} d\hat{s}^2 &= \tilde{\Delta}^{\frac{2}{D-1}} ds_D^2 + \frac{1}{g^2} \tilde{\Delta}^{-\frac{D-3}{D-1}} \sum_i X_i^{-1} d\mu_i^2, \\ \hat{F}_{(D)} &= g \sum_i (2X_i^2 \mu_i^2 - \tilde{\Delta} X_i) \epsilon_{(D)} - \frac{1}{2g} \sum_i X_i^{-1} *dX_i \wedge d(\mu_i^2), \end{aligned} \quad (3.1)$$

where

$$\tilde{\Delta} = \sum_i X_i \mu_i^2, \quad (3.2)$$

and the μ_i are a set of N “direction cosines” that satisfy the constraint

$$\sum_i \mu_i^2 = 1. \quad (3.3)$$

In (3.1), $\epsilon_{(D)}$ denotes the volume form of the D -dimensional metric ds_D^2 .

Clearly, if the scalar fields $\vec{\varphi}$ all vanish, so that we have $X_i = 1$, the internal metric in (3.1) reduces to the standard round metric on the $(N - 1)$ -sphere, proportional to $\sum_i d\mu_i^2$. When the scalars are nonzero, they parameterize inhomogeneous deformations of the round sphere metric. The scalars also appear in the Ansatz for the antisymmetric tensor, as

⁴We place hats on all higher-dimensional fields, and also on the Hodge $*$ operator in the higher dimension.

given in the second line in (3.1). It is straightforward to show that with this Ansatz, the equations of motion and Bianchi identity for the field strength $\hat{F}_{(D)}$ correctly reproduce the D -dimensional scalar equations of motion that we derived in section 2. In fact it is not hard to see that in each of the cases $D = 4, 5$ and 7 , the equations of motion for the scalars come from the Bianchi identity

$$d\hat{F}_{(D)} = 0. \quad (3.4)$$

For the case $D = 4$, (3.4) is, of course, directly the Bianchi identity of the 4-form field of $D = 11$ supergravity. For $D = 7$, where we have $\hat{F}_{(7)} = \hat{*}\hat{F}_{(4)}$, the Bianchi identity (3.4) originates from the equation of motion for $\hat{F}_{(4)}$, namely $d\hat{*}\hat{F}_{(4)} = \frac{1}{2}\hat{F}_{(4)} \wedge \hat{F}_{(4)}$. Since the Ansatz for $\hat{F}_{(7)}$ in (3.1) implies that $\hat{F}_{(4)} \wedge \hat{F}_{(4)} = 0$, we are again just left with (3.4). Finally, in $D = 5$ the Bianchi identity for the self-dual 5-form $\hat{H}_{(5)} = \hat{F}_{(5)} + \hat{*}\hat{F}_{(5)}$ is just $d\hat{H}_{(5)} = 0$, since we are taking the other fields of the type IIB theory to be zero in our reduction. Furthermore, from the form of the Ansatz for $\hat{F}_{(5)}$ given in (3.1) in this case, we can see that we will separately have to have $d\hat{F}_{(5)} = 0$ and $d\hat{*}\hat{F}_{(5)} = 0$, and in fact it is the former that will imply the equations of motion for the X_i . In fact in all three cases, the “field equation” $d\hat{*}\hat{F}_{(D)} = 0$ can be seen to be identically satisfied, supplying no further information.

To see how (3.4) implies the equations of motion for the lower-dimensional scalars, we note from the Ansatz for $\hat{F}_{(D)}$ in (3.1) that the only terms arising will be when the exterior derivative lands on the μ_i coordinates in the prefactor of $\epsilon_{(D)}$, and on the $X_i^{-1} * dX_i$ factor in the last term in the field-strength Ansatz. Thus we deduce that

$$\sum_i \left(\square \log X_i - 4g^2 X_i^2 + 2g^2 X_i \sum_j X_j \right) d(\mu_i^2) = 0. \quad (3.5)$$

We cannot simply deduce from this that the factor enclosed in the large parentheses vanishes for each i , since the μ_i coordinates are not independent, but are subject to the constraint (3.3), implying that $\sum_i d(\mu_i^2) = 0$. Thus the most we can deduce from (3.5) is that

$$\square \log X_i = 4g^2 X_i^2 - 2g^2 X_i \sum_j X_j + Q, \quad (3.6)$$

where Q is an undetermined i -independent quantity. However, since we know that the X_i satisfy (2.3), it follows that $\sum_i \log X_i = 0$. Thus summing over i in (3.6), the left-hand side must vanish, which gives us an equation for Q . Substituting back into (3.6) then gives

$$\square \log X_i = 2g^2 \left(2X_i^2 - X_i \sum_j X_j - \frac{2}{N} \sum_j X_j^2 + \frac{1}{N} \left(\sum_j X_j \right)^2 \right), \quad (3.7)$$

which is precisely the scalar equations of motion that we derived for the Lagrangians in section 2.

A complete verification of the consistency of the reduction Ansatz (3.1) would require substituting it into the higher-dimensional Einstein equation. The components of this equation with indices living in the lower dimensions should then give rise to the D -dimensional Einstein equation (2.22), while the internal and mixed components should give rise to the scalar equations of motion, and to certain non-trivial self-consistency checks. We have not yet performed a complete check of the Einstein equation, on account of the complexity of the curvature calculations that are involved here. However, the computations here are rather similar to ones that arose in [15], where a number of non-trivial checks on the consistency of the Ansatz in the Einstein equation were performed. In fact, if we specialize our Ansätze here by setting as many pairs of X_i 's as possible equal, they can be seen to reduce precisely to special cases of the Ansätze considered in [15].

To see how this occurs, consider first a case where N is even. Then, we may set

$$X_1 = X_2 \equiv \tilde{X}_1, \quad X_3 = X_4 \equiv \tilde{X}_2, \quad \text{etc.} \quad (3.8)$$

and also introduce angles ϕ_a such that

$$\begin{aligned} \mu_1 &= \tilde{\mu}_1 \cos \phi_1, & \mu_2 &= \tilde{\mu}_1 \sin \phi_1, \\ \mu_3 &= \tilde{\mu}_2 \cos \phi_2, & \mu_4 &= \tilde{\mu}_2 \sin \phi_2, \end{aligned} \quad (3.9)$$

etc. (So $\sum_a \tilde{\mu}_a^2 = 1$.) Then, the Ansätze (3.1) become

$$\begin{aligned} d\hat{s}^2 &= \tilde{\Delta}^{\frac{2}{D-1}} ds_D^2 + \frac{1}{g^2} \tilde{\Delta}^{-\frac{D-3}{D-1}} \sum_a \tilde{X}_a^{-1} (d\tilde{\mu}_a^2 + \tilde{\mu}_a^2 d\phi_a^2), \\ \hat{F}_{(D)} &= 2g \sum_a (\tilde{X}_a^2 \tilde{\mu}_a^2 - \tilde{\Delta} \tilde{X}_a) \epsilon_{(D)} - \frac{1}{2g} \sum_a \tilde{X}_a^{-1} *d\tilde{X}_a \wedge d(\tilde{\mu}_a^2), \end{aligned} \quad (3.10)$$

where $\tilde{\Delta}$ is now given by

$$\tilde{\Delta} = \sum_a \tilde{X}_a \tilde{\mu}_a^2, \quad (3.11)$$

and the index a ranges over $1 \leq a \leq \frac{1}{2}N$. Of course the quantities \tilde{X}_a satisfy $\prod_a \tilde{X}_a = 1$, by virtue of (2.3). Equations (3.10) are precisely of the form of the Ansätze obtained in [15] for the cases of the S^7 reduction of $D = 11$ supergravity, and the S^5 reduction of type IIB supergravity, where in each case a truncation to the maximal abelian subgroup of the full gauge group was made. (Of course here, where we are considering just the metric and scalar fields in the lower-dimensional supergravities, the surviving $U(1)^4$ and $U(1)^3$ gauge fields of the maximal abelian truncation are also set to zero.)

In the case when N is odd, a similar type of specialization leads us back to a consistent embedding of the form that was derived in [15] for the maximal abelian $U(1)^2$ truncation of $D = 7$ gauged supergravity into $D = 11$ supergravity compactified on S^4 . In this case, we make pairwise identifications for the first $(N - 1)$ X_i and μ_i , as in (3.8) and (3.9), and keep the single left over quantities $\tilde{X}_0 \equiv X_N$, and $\tilde{\mu}_0 \equiv \mu_N$ as they are. Now, we find that the Asätze (3.1) become

$$\begin{aligned} d\hat{s}^2 &= \tilde{\Delta}^{\frac{2}{D-1}} ds_D^2 + \frac{1}{g^2} \tilde{\Delta}^{-\frac{D-3}{D-1}} \left(\tilde{X}_0^{-1} d\tilde{\mu}_0^2 + \sum_a \tilde{X}_a^{-1} (d\tilde{\mu}_a^2 + \tilde{\mu}_a^2 d\phi_a^2) \right), \\ \hat{F}_{(D)} &= 2g \sum_a (\tilde{X}_a^2 \tilde{\mu}_a^2 - \tilde{\Delta} \tilde{X}_a) \epsilon_{(D)} + g \tilde{\Delta} \tilde{X}_0 \epsilon_{(D)} \\ &\quad - \frac{1}{2g} \tilde{X}_0^{-1} *d\tilde{X}_0 \wedge d(\tilde{\mu}_0^2) - \frac{1}{2g} \sum_a \tilde{X}_a^{-1} *d\tilde{X}_a \wedge d(\tilde{\mu}_a^2), \end{aligned} \quad (3.12)$$

where $\tilde{\Delta}$ is given by

$$\tilde{\Delta} = \tilde{X}_0 \tilde{\mu}_0^2 + \sum_a \tilde{X}_a \tilde{\mu}_a^2, \quad (3.13)$$

and we are taking a to range over $1 \leq \frac{1}{2}(N - 1)$ here. Note that the tilded fields satisfy the constraint $\tilde{X}_0 = \prod_a \tilde{X}_a^{-2}$.

4 Domain Walls as Distributed p -branes

Having obtained the reduction Ansatz for the scalar potentials of lower-dimensional gauged supergravities from Kaluza-Klein sphere reductions, it is straightforward to oxidize the AdS domain wall solutions obtained in section 2.2 to higher dimensions. In this section we show that the AdS₇, AdS₄ and AdS₅ domain wall solutions can be viewed as distributed M5-branes, M2-branes and D3-branes respectively.

4.1 AdS₇ domain walls as distributed M5-branes

Using the reduction Ansatz, we find that the oxidation of the AdS₇ domain wall to $D = 11$ becomes

$$d\hat{s}_{11}^2 = H^{-1/3} (-dt^2 + d\vec{x} \cdot d\vec{x}) + H^{2/3} ds_5^2, \quad \hat{F}_{(4)} = \hat{*}(dt \wedge d^5x \wedge dH^{-1}), \quad (4.1)$$

where

$$H = \frac{1}{g^3 r^3 \Delta} \quad \Delta = (H_1 \cdots H_5)^{1/2} \sum_i \frac{\mu_i^2}{H_i}. \quad (4.2)$$

The transverse-space metric ds_5^2 is given by

$$ds_5^2 = \frac{\Delta dr^2}{\sqrt{H_1 \cdots H_5}} + r^2 \sum_i H_i d\mu_i^2, \quad (4.3)$$

which is in fact a flat Euclidean metric $ds_5^2 = dy^m dy^m$ after making the coordinate transformation

$$y_i = r\sqrt{H_i}\mu_i. \quad (4.4)$$

The solution (4.1) describes a continuous distribution of M5-branes, and the harmonic function H can be expressed as

$$H = g^{-3} \int \frac{\sigma(\vec{y}') d^5 y'}{|\vec{y} - \vec{y}'|^3}. \quad (4.5)$$

Various cases arise, depending on how many of the ℓ_i parameters are non-zero. Without loss of generality, we may consider the cases where the first n parameters ℓ_i^2 are non-vanishing, for each n in the range $0 \leq n \leq 4$. (Note that there is no need to consider the case with 5 non-vanishing parameters since, as we showed previously, this is a degenerate situation that is equivalent to having just 4 non-vanishing parameters.) We find that the charge distribution functions σ in the various cases are given by

$$\begin{aligned} n=0: \quad & \sigma = \delta^{(5)}(\vec{y}), \\ n=1: \quad & \sigma = \frac{1}{2\ell_1} \Theta(1 - \frac{y_1^2}{\ell_1^2}) \delta^{(4)}(\vec{y}_{2,3,4,5}), \\ n=2: \quad & \sigma = \frac{1}{2\pi\ell_1\ell_2} (1 - \frac{y_1^2}{\ell_1^2} - \frac{y_2^2}{\ell_2^2})^{-1/2} \Theta(1 - \frac{y_1^2}{\ell_1^2} - \frac{y_2^2}{\ell_2^2}) \delta^{(3)}(\vec{y}_{3,4,5}), \\ n=3: \quad & \sigma = \frac{1}{2\pi\ell_1\ell_2\ell_3} \delta(1 - \sum_{i=1}^3 \frac{y_i^2}{\ell_i^2}) \delta^{(2)}(\vec{y}_{4,5}), \\ n=4: \quad & \sigma = \frac{1}{4\pi^2\ell_1\cdots\ell_4} \left[- (1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2})^{-3/2} \Theta(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}) \right. \\ & \quad \left. + 2(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2})^{-1/2} \delta(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}) \right] \delta(y_5). \end{aligned} \quad (4.6)$$

The distribution for $n=2$ with $\ell_1 = \ell_2$ were also given in [16]. We may note an interesting relation among these distributions, which will persist in future sections and was first noticed in [7]. If the $n=4$ distribution is integrated along one of its principle axes (that is, in the direction of one of the y_i), and a factor of $\delta(y_i)$ is then inserted, the result is the $n=3$ distribution. And if the $n=3$ distribution is integrated similarly, the result is the $n=2$ distribution—et cetera. As we explained earlier, there is *a priori* no preferred choice of five-parameter solution: for example, if all the ℓ_i were equal, there would be no preferred spherically symmetric distribution of branes since all of them lead to the same perfect $\text{AdS}_7 \times S^4$ exterior. (This is because the five-parameter solution is degenerate owing to the symmetry of the solution given by (2.27).)

However, there is a unique five-parameter solution which shares the property that integrating it along some principle axis leads to the $n = 4$ solution. It is

$$n = 5 : \quad \sigma = \frac{1}{2\pi^2 \ell_1 \ell_2 \ell_3 \ell_4 \ell_5} \delta' \left(1 - \sum_{i=1}^5 \frac{y_i^2}{\ell_i^2} \right). \quad (4.7)$$

All the other distributions, (4.6), can be regarded as limits of (4.7) where some of the ℓ_i are taken to 0. Such a limiting process is equivalent to integrating along the corresponding y_i directions and then inserting factors of $\delta(y_i)$.

The above discussion is applicable for positive values of ℓ_i^2 . The solutions with negative ℓ_i^2 can be mapped to the solutions with positive ℓ_i^2 using the transformation (2.27), as discussed in section 2.2.

4.2 AdS₄ domain walls as distributed M2-branes

Analogously, we find that the oxidation of the AdS₄ domain wall to $D = 11$ becomes

$$d\hat{s}_{11}^2 = H^{-2/3} (-dt^2 + d\vec{x} \cdot d\vec{x}) + H^{1/3} ds_8^2, \quad \hat{F}_{(4)} = dt \wedge d^2x \wedge dH^{-1}, \quad (4.8)$$

where

$$H = \frac{1}{g^6 r^6 \Delta} \quad \Delta = (H_1 \cdots H_8)^{1/2} \sum_{i=1}^8 \frac{\mu_i^2}{H_i}. \quad (4.9)$$

The transverse-space metric ds_8^2 is given by

$$ds_8^2 = \frac{\Delta dr^2}{\sqrt{H_1 \cdots H_8}} + r^2 \sum_i H_i d\mu_i^2, \quad (4.10)$$

which is in fact a flat Euclidean 8-metric $ds_8^2 = dy^m dy^m$ after making the coordinate transformation

$$y_i = r \sqrt{H_i} \mu_i. \quad (4.11)$$

The solution (4.8) describes a continuous distribution of M2-branes, and the harmonic function H can be expressed as

$$H = g^{-6} \int \frac{\sigma(\vec{y}') d^8 y'}{|\vec{y} - \vec{y}'|^6}. \quad (4.12)$$

For the various cases with $0 \leq n \leq 7$ parameters ℓ_i , we find that the charge distribution functions σ are given by

$$\begin{aligned} n = 0 : \quad & \sigma = \delta^{(8)}(\vec{y}), \\ n = 1 : \quad & \sigma = \frac{8}{3\pi \ell_1} \left(1 - \frac{y_1^2}{\ell_1^2} \right)^{3/2} \Theta \left(1 - \frac{y_1^2}{\ell_1^2} \right) \delta^{(7)}(\vec{y}_{2,3,4,5,6,7,8}), \end{aligned}$$

$$\begin{aligned}
n=2: \quad \sigma &= \frac{2}{\pi \ell_1 \ell_2} \left(1 - \frac{y_1^2}{\ell_1^2} - \frac{y_2^2}{\ell_2^2}\right) \Theta\left(1 - \frac{y_1^2}{\ell_1^2} - \frac{y_2^2}{\ell_2^2}\right) \delta^{(6)}(\vec{y}_{3,4,5,6,7,8}), \\
n=3: \quad \sigma &= \frac{4}{\pi^2 \ell_1 \ell_2 \ell_3} \left(1 - \sum_{i=1}^3 \frac{y_i^2}{\ell_i^2}\right)^{1/2} \Theta\left(1 - \sum_{i=1}^3 \frac{y_i^2}{\ell_i^2}\right) \delta^{(5)}(\vec{y}_{4,5,6,7,8}), \\
n=4: \quad \sigma &= \frac{2}{\pi^2 \ell_1 \dots \ell_4} \Theta\left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}\right) \delta^{(4)}(\vec{y}_{5,6,7,8}), \\
n=5: \quad \sigma &= \frac{2}{\pi^3 \ell_1 \dots \ell_5} \left(1 - \sum_{i=1}^5 \frac{y_i^2}{\ell_i^2}\right)^{-1/2} \Theta\left(1 - \sum_{i=1}^5 \frac{y_i^2}{\ell_i^2}\right) \delta^{(3)}(\vec{y}_{6,7,8}), \\
n=6: \quad \sigma &= \frac{2}{\pi^3 \ell_1 \dots \ell_6} \delta\left(1 - \sum_{i=1}^6 \frac{y_i^2}{\ell_i^2}\right) \delta^{(2)}(\vec{y}_{7,8}), \\
n=7: \quad \sigma &= \frac{1}{\pi^4 \ell_1 \dots \ell_7} \left[-\left(1 - \sum_{i=1}^7 \frac{y_i^2}{\ell_i^2}\right)^{-3/2} \Theta\left(1 - \sum_{i=1}^7 \frac{y_i^2}{\ell_i^2}\right) \right. \\
&\quad \left. + 2\left(1 - \sum_{i=1}^7 \frac{y_i^2}{\ell_i^2}\right)^{-1/2} \delta\left(1 - \sum_{i=1}^7 \frac{y_i^2}{\ell_i^2}\right) \right] \delta(y_8).
\end{aligned} \tag{4.13}$$

The distribution for $n=2$ with $\ell_1 = \ell_2$ was also given in [16]. Note that the 8-parameter case is degenerate, since it is equivalent to a 7-parameter solution. However, as in the case of M5-branes, the distribution

$$n=8: \quad \sigma = \frac{2}{\pi^4 \ell_1 \dots \ell_8} \delta'\left(1 - \sum_{i=1}^8 \frac{y_i^2}{\ell_i^2}\right) \tag{4.14}$$

can be regarded as the “parent” of all the others in the sense that all the distributions in (4.13) are limits of (4.14) where some of the ℓ_i are taken to 0.

The discussion is applicable for positive values of ℓ_i^2 . The solutions with negative ℓ_i^2 can be mapped to the solutions with positive ℓ_i^2 using the transformation (2.27), as discussed in section 2.2.

4.3 AdS₅ domain walls as distributed D3-branes

Analogously, we find that the oxidation of the AdS₅ domain wall to $D=10$ becomes

$$d\hat{s}_{10}^2 = H^{-1/2}(-dt^2 + d\vec{x} \cdot d\vec{x}) + H^{1/2} ds_6^2, \quad \hat{G}_{(5)} = dt \wedge d^3x \wedge dH^{-1}, \tag{4.15}$$

where

$$H = \frac{1}{g^4 r^4 \Delta} \quad \Delta = (H_1 \dots H_6)^{1/2} \sum_{i=1}^6 \frac{\mu_i^2}{H_i}. \tag{4.16}$$

The transverse-space metric ds_6^2 is given by

$$ds_6^2 = \frac{\Delta dr^2}{\sqrt{H_1 \dots H_6}} + r^2 \sum_i H_i d\mu_i^2, \tag{4.17}$$

which is in fact a flat Euclidean 6-metric $ds_6^2 = dy^m dy^m$ after making the coordinate transformation

$$y_i = r\sqrt{H_i}\mu_i. \quad (4.18)$$

The solution (4.15) describes a continuous distribution of D3-branes, and the harmonic function H can be expressed as

$$H = g^{-4} \int \frac{\sigma(\vec{y}') d^8 y'}{|\vec{y} - \vec{y}'|^4}. \quad (4.19)$$

For the various $0 \leq n \leq 4$ parameter configurations, we find that the charge distribution functions σ are given by

$$\begin{aligned} n=0: \quad \sigma &= \delta^{(6)}(\vec{y}), \\ n=1: \quad \sigma &= \frac{2}{\pi\ell_1} \left(1 - \frac{y_1^2}{\ell_1^2}\right)^{1/2} \Theta\left(1 - \frac{y_1^2}{\ell_1^2}\right) \delta^{(5)}(\vec{y}_{2,3,4,5,6}), \\ n=2: \quad \sigma &= \frac{1}{\pi\ell_1\ell_2} \Theta\left(1 - \frac{y_1^2}{\ell_1^2} - \frac{y_2^2}{\ell_2^2}\right) \delta^{(4)}(\vec{y}_{3,4,5,6}), \\ n=3: \quad \sigma &= \frac{1}{\pi^2\ell_1\ell_2\ell_3} \left(1 - \sum_{i=1}^3 \frac{y_i^2}{\ell_i^2}\right)^{-1/2} \Theta\left(1 - \sum_{i=1}^3 \frac{y_i^2}{\ell_i^2}\right) \delta^{(3)}(\vec{y}_{4,5,6}), \\ n=4: \quad \sigma &= \frac{1}{\pi^2\ell_1 \dots \ell_4} \delta\left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}\right) \delta^{(2)}(\vec{y}_{5,6}), \\ n=5: \quad \sigma &= \frac{1}{2\pi^3\ell_1 \dots \ell_5} \left[-\left(1 - \sum_{i=1}^5 \frac{y_i^2}{\ell_i^2}\right)^{-3/2} \Theta\left(1 - \sum_{i=1}^5 \frac{y_i^2}{\ell_i^2}\right) \right. \\ &\quad \left. + 2\left(1 - \sum_{i=1}^5 \frac{y_i^2}{\ell_i^2}\right)^{-1/2} \delta\left(1 - \sum_{i=1}^5 \frac{y_i^2}{\ell_i^2}\right) \right] \delta(y_6). \end{aligned} \quad (4.20)$$

Various special cases of the above list can also be found in [16, 7]. Note that the 6-parameter case is degenerate, being equivalent to a 5-parameter solution through the transformation (2.27). Once again, though, there is a unique 6-parameter distribution,

$$n=6: \quad \sigma = \frac{1}{\pi^3\ell_1 \dots \ell_6} \delta'\left(1 - \sum_{i=1}^6 \frac{y_i^2}{\ell_i^2}\right) \quad (4.21)$$

such that all the other distributions (4.20) are limits of (4.21) in which some of the ℓ_i are taken to 0.

The above discussions are applicable for positive values of ℓ_i^2 . The solutions with negative ℓ_i^2 can be mapped to the solutions with positive ℓ_i^2 using the transformation (2.27), as discussed in section 2.2.

Note that the charge distributions $\sigma(\vec{y})$ have the same functional form for M2-branes, M5-branes and D3-branes with the same $N - n$.

5 Six-dimensional Gauged Supergravity

In our discussion so far, we have focussed on the class of scalar plus gravity theories that correspond to certain consistent truncations of the gauged maximal supergravities in $D = 4$, 5 and 7. There is one other case that we wish to consider here, namely gauged supergravity in $D = 6$. This is an unusual case in that there apparently does not exist a gauged supergravity with the maximum number of supersymmetries that are allowed in six dimensions. The known $SU(2)$ gauged theory has $N = 2$ rather than $N = 4$ supersymmetry [24]. It was suggested in [25] that it could be related to the ten-dimensional massive type IIA theory [26]. It has recently been shown that it can be obtained from an S^4 reduction of massive type IIA supergravity [12]. Since it does not have maximal supersymmetry there is nothing to preclude coupling it to $N = 2$ matter. The six-dimensional scalar plus gravity theory that we shall consider here can be interpreted as a consistent truncation of the $SU(2)$ gauged $N = 2$ supergravity in $D = 6$ coupled to certain matter multiplets. (Domain wall solutions in pure $D = 6$ gauged supergravity were previously studied in [27, 28].

The only part of the massive IIA Lagrangian that is relevant for our truncation comprises the metric, the dilaton and the 4-form field strength:

$$\mathcal{L}_{10} = \hat{R} * \mathbf{1} - \frac{1}{2} * d\hat{\phi} \wedge d\hat{\phi} - \frac{1}{2} e^{\frac{1}{2}\hat{\phi}} * \hat{F}_{(4)} \wedge \hat{F}_{(4)} - \frac{1}{2} m^2 e^{\frac{5}{2}\hat{\phi}} * \mathbf{1}. \quad (5.1)$$

This admits a solution describing an intersection of D4-branes and D8-branes, namely [29]

$$\begin{aligned} ds_{10}^2 &= (g z)^{\frac{1}{12}} \left(H^{-3/8} (-dt^2 + d\vec{x} \cdot d\vec{x}) + H^{5/8} (d\vec{y}^2 + dz^2) \right), \\ e^{\hat{\phi}} &= (g z)^{-\frac{5}{6}} H^{-1/4} \quad F_4 = e^{-\frac{1}{2}\hat{\phi}} * (d^5 x \wedge dH^{-1}), \end{aligned} \quad (5.2)$$

where $g = 3m/\sqrt{2}$ and the “harmonic” function satisfies

$$z^{-1/3} \partial_z (z^{1/3} \partial_z H) + \square_y H = 0. \quad (5.3)$$

One class of solutions of this differential equation can be expressed as follows [29]:

$$H = c + g^{-10/3} \int \frac{\sigma(\vec{y}') d^4 y'}{(|\vec{y} - \vec{y}'|^2 + z^2)^{5/3}}, \quad (5.4)$$

where c is an integration constant, which we shall set to zero. It should be emphasized, however, that this is not the most general solution to equation (5.3). If the distribution function σ is a delta function, *i.e.* $\sigma = \delta^{(4)}(\vec{y})$, the metric becomes warped $\text{AdS}_6 \times S^4$ [30].

Inspired by the previous examples of M-branes and the D3-brane, let us consider the following charge distribution,

$$\sigma = \frac{2}{9\pi^2 \ell_1 \dots \ell_4} \left[- \left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2} \right)^{-4/3} \Theta \left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2} \right) \right]$$

$$+3\left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}\right)^{-1/3} \delta\left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}\right) \Big]. \quad (5.5)$$

To evaluate H with this distribution, we perform the following coordinate transformation:

$$y_i = \sqrt{r^2 + \ell_i^2} \mu_i, \quad z = \sqrt{r^2 + \ell_0^2} \mu_0, \quad (5.6)$$

where $\sum_\alpha \mu_\alpha^2 = 1$, summing over $\alpha = (0, i)$. Note, however, that although we included a parameter ℓ_0 for the sake of symmetry, it should actually be set to zero. Next, we define $H_\alpha = 1 + \ell_\alpha^2/r^2$ (and so $H_0 = 1$). It then follows that the function H becomes

$$H = \frac{1}{(gr)^{10/3} \Delta}, \quad \Delta = \left(\prod_\alpha H_\alpha\right)^{1/2} \sum_\alpha \frac{\mu_\alpha^2}{H_\alpha}. \quad (5.7)$$

In terms of the redefined coordinates y^i and z , the transverse-space metric is

$$dy^i dy^i + dz^2 = \left(\prod_\alpha H_\alpha\right)^{-1/2} \Delta dr^2 + r^2 H_\alpha d\mu_\alpha^2. \quad (5.8)$$

Thus the ten-dimensional metric of the D4-D8-brane becomes

$$ds_{10}^2 = \mu_0^{1/12} \left[(gr)^{4/3} \Delta^{3/8} dx^\mu dx_\mu + \Delta^{3/8} \left(\prod_\alpha H_\alpha\right)^{-1/2} \frac{dr^2}{g^2 r^2} + \Delta^{-5/8} g^{-2} \sum_\alpha H_\alpha d\mu_\alpha^2 \right]. \quad (5.9)$$

If we now define

$$\tilde{\Delta} = \left(\prod_\alpha H_\alpha\right)^{-5/16} \Delta = \left(\prod_\alpha H_\alpha\right)^{3/16} \sum_\alpha \frac{\mu_\alpha^2}{H_\alpha} \equiv \sum_\alpha X_\alpha \mu_\alpha^2, \quad (5.10)$$

the ten-dimensional D4-D8-brane solution can be expressed in the following abstract form:

$$\begin{aligned} ds_{10}^2 &= \mu_0^{1/12} \left(\prod_\alpha X_\alpha\right)^{1/8} \left(\tilde{\Delta}^{3/8} ds_6^2 + g^{-2} \tilde{\Delta}^{-5/8} \sum_\alpha X_\alpha^{-1} d\mu_\alpha^2 \right), \\ e^{\frac{1}{2}\hat{\phi}} * \hat{F}_{(4)} &= g \sum_\alpha (2X_\alpha^2 \mu_\alpha^2 - \tilde{\Delta} X_\alpha) \epsilon_{(6)} - \frac{1}{3} g \tilde{\Delta} X_0 \epsilon_{(6)} + \frac{1}{2g} \sum_\alpha X_\alpha^{-1} * dX_\alpha d(\mu_\alpha^2), \\ e^{\hat{\phi}} &= \mu_0^{-5/16} \tilde{\Delta}^{1/4} \left(\prod_\alpha X_\alpha\right)^{-5/4}, \end{aligned} \quad (5.11)$$

where the six-dimensional metric and scalars X_α are given by

$$\begin{aligned} ds_6^2 &= (gr)^{4/3} (H_1 \cdots H_4)^{1/8} dx^\mu dx_\mu + (H_1 \cdots H_4)^{-3/8} \frac{dr^2}{g^2 r^2}, \\ X_i &= (H_1 \cdots H_4)^{3/16} H_i^{-1}, \quad X_0 = (H_1 \cdots H_4)^{3/16}. \end{aligned} \quad (5.12)$$

Note that the quantities X_α are not independent, but are subject to the constraint

$$X_0 = (X_1 \cdots X_4)^{-3/4}. \quad (5.13)$$

Having written the solutions in an abstract form, we can now propose to view (5.11) as a general Kaluza-Klein reduction Ansatz, for arbitrary six-dimensional fields, whose equations of motion can be derived by substituting the Ansatz into the ten-dimensional equations of motion for the massive IIA theory. (Of course, for this to succeed, it is necessary that the reduction Ansatz (5.11) be a consistent one, which appears to be the case.) In particular, we can determine the structure of the scalar potential. This is described by four independent scalars, parameterised by X_α , subject to the constraint (5.13).

5.1 $D = 6$ potential

We can work this out by starting from the Ansatz (5.11) for the 4-form field strength. Its Bianchi identity gives us the equations of motion for the X_α . Thus we can immediately read off that

$$\square \log X_\alpha = 4g^2 X_\alpha^2 - 2g^2 X_\alpha \sum_\beta X_\beta - \frac{2}{3}g^2 X_0 X_\alpha + Q, \quad (5.14)$$

where Q is an undetermined quantity independent of α . (The ambiguity here arises, as usual, because $\sum_\alpha d(\mu_\alpha^2) = 0$, by virtue of $\sum_\alpha \mu_\alpha^2 = 1$.) We can determine Q by noting that $X_0 = (\prod_i X_i)^{-3/4}$, and hence

$$\frac{4}{3} \log X_0 + \sum_i \log X_i = 0. \quad (5.15)$$

(Recall we are splitting the indices as $\alpha = (0, i)$.) Thus if we take this particular sum over terms in (5.14), the left-hand side must vanish, hence giving us an equation for Q :

$$Q = g^2 X_0 \sum_i X_i - \frac{1}{3}g^2 X_0^2 - \frac{3}{4}g^2 \sum_i X_i^2 + \frac{3}{8}g^2 (\sum_i X_i)^2. \quad (5.16)$$

In particular, this means, plugging back into (5.14), that we have

$$\begin{aligned} \square \log X_0 &= g^2 X_0^2 - g^2 X_0 \sum_i X_i - \frac{3}{4}g^2 \sum_i X_i^2 + \frac{3}{8}g^2 (\sum_i X_i)^2, \\ \square \log X_i &= 4g^2 X_i^2 - 2g^2 X_i \sum_j X_j - \frac{8}{3}g^2 X_0 X_i \\ &\quad + \frac{3}{8}g^2 (\sum_j X_j)^2 - \frac{3}{4}g^2 \sum_j X_j^2 + g^2 X_0 \sum_j X_j - \frac{1}{3}g^2 X_0^2. \end{aligned} \quad (5.17)$$

We now look for a 6-dimensional potential V such that

$$e^{-1} \mathcal{L}_6 = R - \frac{1}{2}(\partial \vec{\varphi})^2 - V \quad (5.18)$$

reproduces the above equations of motion, where $\vec{\varphi}$ is a 4-vector of dilatons. We express X_0 and X_i in terms of these in the usual way, $X_\alpha = e^{-\frac{1}{2}\vec{b}_\alpha \cdot \vec{\varphi}}$. At this stage, let us just present

the results. We find that to reproduce the equations of motion (5.17) correctly, we should take the vectors \vec{b}_α to satisfy the dot products

$$\vec{b}_i \cdot \vec{b}_j = 8\delta_{ij} - \frac{3}{2}, \quad \vec{b}_0 \cdot \vec{b}_i = -\frac{3}{2}, \quad \vec{b}_0 \cdot \vec{b}_0 = \frac{9}{2}. \quad (5.19)$$

Of course the identity $X_0 = (\prod_i X_i)^{-3/4}$ implies that we will also have

$$\vec{b}_0 + \frac{3}{4} \sum_i \vec{b}_i = 0. \quad (5.20)$$

The required scalar potential then turns out to be

$$V = -\frac{1}{2}g^2 \left(\left(\sum_i X_i \right)^2 - 2 \sum_i X_i^2 + \frac{8}{3} X_0 \sum_i X_i - \frac{8}{9} X_0^2 \right). \quad (5.21)$$

Note that if we specialise to the case where $X_1 = X_2 \equiv \tilde{X}_1$, $X_3 = X_4 \equiv \tilde{X}_2$, $X_0 \equiv \tilde{X}_0$, we obtain the potential

$$V = \frac{4}{9}g^2 (\tilde{X}_0^2 - 9\tilde{X}_1 \tilde{X}_2 - 6\tilde{X}_0 \tilde{X}_1 - 6\tilde{X}_0 \tilde{X}_2). \quad (5.22)$$

Note that a further specialisation $\tilde{X}_1 = \tilde{X}_2$ gives back the potential of $D = 6$ gauged supergravity, obtained from massive type IIA in [12].

5.2 AdS₆ domain walls as distributed D4-D8-branes

The scalar Lagrangian (5.18) with potential (5.21) admits a six-dimensional domain wall solution with four parameters ℓ_i , given by (5.12). From the $D = 10$ standpoint, the solution can be viewed as a distribution of D4-D8-branes with charge distribution function (5.5). For the various numbers of non-vanishing parameters ℓ_i , the charge distribution function σ is given by

$$\begin{aligned} n=0: \quad & \sigma = \delta^{(4)}(\vec{y}), \\ n=1: \quad & \sigma = \frac{\Gamma(5/3)}{\sqrt{\pi} \Gamma(7/6) \ell_1} \left(1 - \frac{y_1^2}{\ell_1^2}\right)^{1/6} \Theta\left(1 - \frac{y_1^2}{\ell_1^2}\right) \delta^{(3)}(\vec{y}_{2,3,4}), \\ n=2: \quad & \frac{2}{3\pi \ell_1 \ell_2} \left(1 - \frac{y_1^2}{\ell_1^2} - \frac{y_2^2}{\ell_2^2}\right)^{-1/3} \Theta\left(1 - \frac{y_1^2}{\ell_1^2} - \frac{y_2^2}{\ell_2^2}\right) \delta^{(2)}(\vec{y}_{3,4}), \\ n=3: \quad & \frac{\Gamma(5/3)}{\pi^{3/2} \Gamma(1/6) \ell_1 \ell_2 \ell_3} \left(1 - \sum_{i=1}^3 \frac{y_i^2}{\ell_i^2}\right)^{-5/6} \Theta\left(1 - \sum_{i=1}^3 \frac{y_i^2}{\ell_i^2}\right) \delta(y_4), \\ n=4: \quad & \sigma = \frac{2}{9\pi^2 \ell_1 \dots \ell_4} \left[- \left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}\right)^{-4/3} \Theta\left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}\right) \right. \\ & \quad \left. + 3 \left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}\right)^{-1/3} \delta\left(1 - \sum_{i=1}^4 \frac{y_i^2}{\ell_i^2}\right) \right]. \end{aligned} \quad (5.23)$$

A remark about the evaluation of the integral (5.4) with the charge distributions (5.23) is in order here. In the previous cases, of the D3-brane, M2-brane and M5-brane, the analogous integrals could be studied rather easily (as in [16]) by first specialising the coordinates \vec{y} in the resulting harmonic functions $H(\vec{y})$ to simple values for which the integration is more easily performed. $H(\vec{y})$ for this specialised range of \vec{y} coordinate values can then be matched to the general solution of Laplace's equation, expressed as a sum over a complete set of elementary solutions, yielding a unique determination of the expansion coefficients. This then establishes that provided $H(\vec{y})$ evaluated over the specialised coordinate range agrees with the claimed general form for H evaluated over the same range, then the integral over the distribution must be in agreement with the claimed expression for $H(\vec{y})$ for arbitrary coordinate values \vec{y} . Thus, for example, when evaluating (4.5) for the $\ell_i = (\ell_1, \ell_2, 0, 0, 0)$ charge distribution given in (4.6), it suffices to evaluate the integral for $H(\vec{y})$ at $y_1 = y_2 = 0$, and verify that it agrees with H given by (4.2) in this restricted coordinate region. One can, of course, instead directly evaluate the integral (4.5) for $H(\vec{y})$ with generic values for \vec{y} , but this is quite a bit more complicated in practice. In our present case, with the evaluation of (5.4) for the D4/D8-brane distributions (5.23), one is on less solid ground if one tries to invoke uniqueness to argue that it suffices to evaluate the integral for $H(\vec{y})$ at special values of \vec{y} , since, as we remarked, (5.4) is not the most general solution of the equations for D4/D8-brane intersections. We have therefore performed explicit integrations for generic \vec{y} coordinate values, in order to verify that the distribution integrals do indeed correctly reproduce the functions given by (5.7).

The discussion is applicable only for non-negative ℓ_i^2 . For solutions with negative ℓ_i^2 , it can no longer be viewed as distributions of solutions of the form (5.4). This is not surprising, since unlike the D3-brane and M-branes, the harmonic function here (5.4) is not the most general solution of its equation (5.3). Thus whilst for the previous cases the solutions have to be expressible in terms of distributions of D3-branes or M-branes, here it is more of a “miracle” that the six-dimensional AdS domain walls with positive ℓ_i^2 can be expressed as distributions of D4-D8-branes, with the form (5.4).

It is worth remarking that the D4-D8-brane system also has other features that are rather different from those in the D3-brane, M2-brane and M5-brane systems. The distributed D3-branes, M2-branes and M5-branes in $D = 10$ or $D = 11$ admit lower-dimensional interpretations as extremal solutions in the gauged maximal supergravities in $D = 5$, $D = 4$ and $D = 7$ respectively. By contrast, the D4-D8-brane systems admit a lower-dimensional interpretation in $D = 6$, where there apparently does not exist any maximally-supersymmetric

gauged supergravity. There does exist a gauged theory with one half of the maximal supersymmetry in $D = 6$ [24] (namely $N = 2$), and an $SU(2)$ gauge group, which arises as a consistent S^4 reduction of massive type IIA supergravity [12]. However, this theory contains only a single scalar field X , which corresponds to the truncation $X_1 = X_2 = X_3 = X_4 \equiv X$, $X_0 = X^{-3}$ in the six-dimensional potential (5.21). Thus all four of the original ℓ_i parameters in the solutions (5.12) must be set equal in order that the configuration be a solution purely within the $SU(2)$ -gauged $N = 2$ supersymmetric theory. We have shown that the more general configurations (5.12) with all four ℓ_i different are solutions of the six-dimensional theory (5.18) and (5.21), which can also be obtained as a consistent dimensional reduction of the massive type IIA supergravity. However, this six-dimensional theory is not a truncation of the pure gauged supergravity in $D = 6$; rather, it is a truncation of a gauged supergravity coupled to matter.⁵

6 Analysis of the spectrum

The correlations functions (and the spectrum) of the corresponding strongly coupled gauge theory can be analyzed by studying the wave equations in these gravitational backgrounds. In particular, the simplest two-point function is that of the operator $\mathcal{O} \sim \text{Tr} F^2 + \dots$ which couples to the s -wave dilaton ϕ . In ten dimensions the dilaton obeys a free wave equation. In lower dimensions, a consistent truncation ensures that the dilaton can be taken to be independent of the spherical coordinates, and the s -wave dilaton does not participate in the gauged supergravity scalar potential. Generically we expect there always to be a minimal scalar in our asymptotically AdS geometries: a field ϕ which obeys the corresponding Laplace equation,

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0. \quad (6.1)$$

We choose the Ansatz $\phi = e^{i\omega t} \chi(r)$, where ω characterizes the energy level of the solution, specified by $\chi(r)$, which is chosen to depend only on the radial coordinate r . Then for the distributed D3-branes and M-branes discussed in the previous sections, the wave equation

⁵A closely parallel situation which, however, highlights the distinction is that of the two-charge AdS black hole solutions of maximal ($N = 2$) $SO(5)$ -gauged supergravity in $D = 7$. If the charges are set equal, the black holes can be viewed as solutions in pure $N = 1$ $SU(2)$ -gauged supergravity, but if the charges are unequal, then from an $N = 1$ standpoint the black holes are solutions of $N = 1$ gauged supergravity coupled to the matter that results from viewing the $N = 2$ pure supergravity as an $N = 1$ theory. The difference in $D = 6$ is that there *is* no maximal gauged supergravity that could provide the alternative “ $N = 2$ viewpoint.”

(6.1) takes the following form:⁶

$$r^{-1}\partial_r \left[r^{-1} \prod_{i=1}^N \sqrt{r^2 + \ell_i^2} \partial_r \chi \right] = \mathcal{Q} \chi, \quad (6.2)$$

where $\mathcal{Q} = -\omega^2/g^{N/2}$ and $N = 8, 6$ and 5 , for the M2-brane, D3-brane and M5-brane cases, respectively. The wave equation for the distributed D4-D8-brane cases has a somewhat different form, given by

$$r^{-1}\partial_r \left[r^{1/3} \prod_{i=1}^4 \sqrt{r^2 + \ell_i^2} \partial_r \chi \right] = \mathcal{Q} \chi. \quad (6.3)$$

Here $\mathcal{Q} = -\omega^2/g^{10/3}$.

6.1 Spectrum with “Euclidean” parameters

In the approach to modelling confining gauge theories through higher dimensional branes at finite temperature (initiated in [35]), the spectrum of low-energy excitations in the gauge theory is the same as the spectrum of supergravity modes in the dual asymptotically anti-de Sitter geometry. By solving the eigenvalue problems (6.2) and (6.3) rotated to Euclidean signature we are gleaning information regarding the dual gauge theory. Although it has been argued in [31] that some of these gauge theories are indeed confining, the analysis of [34] suggests that often they are not. The issue raised in [34] is that the ratio of the tree-level thermal mass to the temperature vanishes for some of the gauginos in a limit corresponding to one of the domain walls we have considered in this paper. The would-be confining gauge theory approaches a non-confining supersymmetric theory in the limit. What sets the mass scale of the spectrum in such a case is not the scale of confinement, but rather the Higgs VEV that determines the state on the Coulomb branch. We will see from supergravity calculations that the spectrum can be discrete, or a continuum above a gap, or a continuum with no gap. The field theory understanding of these results is not very satisfactory, although some attempts at explanation have been made in [7, 21, 22, 23] (mainly in the Lorentzian cases).

When Wick-rotating (6.2) and (6.3) to Euclidean signature, we have a choice of whether to change the signs of the quantities ℓ_i^2 . The relationship of some of these geometries to spinning brane geometries [32, 33] suggests that the natural choice is to send $\ell_i^2 \rightarrow -\ell_i^2$ in the Wick rotation. The reason is that the ℓ_i are proportional to the angular momenta in the

⁶The glue-ball (0^{++}) spectra for the Euclidean spinning brane backgrounds are determined by the same wave equations and thus the same analysis is applicable in these cases also. (See [34] and references therein.)

spinning brane geometries, and angular momentum becomes imaginary on Wick rotation. Thus we will refer to $\ell_i^2 < 0$ as a choice of Euclidean parameters. For M2-branes, D3-branes, and M5-branes, the transformation (2.27) allows us to map Euclidean parameters back to Lorentzian parameters. The only Lorentzian cases that need to be addressed separately are for the D4-D8-brane system; see section 6.2.

The wave equations (6.2, 6.3) can be cast in the following form:

$$[\partial_y f(y) \partial_y - \mathcal{Q}] \chi = 0, \quad (6.4)$$

where $y \equiv r^2 - r_H^2$ and r_H^2 is an adjustable parameter. For M2-branes, D3-branes, and M5-branes, we have

$$f(y) = \sqrt{\prod_{i=1}^N (y + r_H^2 - \ell_i^2)},$$

where as usual $N = 8, 6$, and 5 in the respective cases. Let us choose r_H^2 so that the most positive root of $f(y)$ is at $y = 0$. We will solve the equation (6.4) subject to regular boundary conditions at $y = 0$ and $y = \infty$, which is the boundary of AdS. We are thus excluding from consideration geometries like spherical shells of branes: the shell would sit at a finite value of y where the norm of the Killing vector $\partial/\partial t$ is finite. To put it another way, we set one of the ℓ_i to 0 so that we have only $N - 1$ free parameters. Generically, (6.4) cannot be solved exactly (although in special cases it reduces to a form of the hypergeometric equation [7, 21]). However we may straightforwardly extract the qualitative features of the spectrum by analyzing the asymptotics of (6.4) near the horizon ($y \rightarrow 0$) and near the boundary of AdS ($y \rightarrow \infty$). The qualitative features depend only on how many of the parameters ℓ_i are nonzero. Let us take $\ell_i^2 = -1$ for n_e values of i , and set $\ell_i^2 = 0$ for the other $N - n_e$ values. Then the function $f(y)$ takes the form:

$$f(y) = 4y^{\frac{1}{2}n_e} (y + 1)^{\frac{1}{2}(N - n_e)}, \quad (6.5)$$

with $N = 8, 6, \frac{16}{3}$ and 5 for the M2-brane, D3-brane, D4-D8-brane and M5-brane cases respectively. Note that $f(y)$ is an increasing positive function of y with the asymptotic behavior

$$\begin{aligned} f(y) &\sim y^{\frac{1}{2}N} && \text{for } y \rightarrow \infty, \\ f(y) &\sim y^{\frac{1}{2}n_e} && \text{for } y \rightarrow 0. \end{aligned} \quad (6.6)$$

For the analysis of the spectrum it is useful to cast the wave equations (6.4) into the form of the Schrödinger equation. This is accomplished by introducing the new coordinate

z , defined by $\partial y / \partial z = \sqrt{f(y)}$, and setting $\chi = \psi f^{-1/4}$. We then find that (6.4) indeed takes the form of Schrödinger equation:

$$\begin{aligned} \left[-\partial_z^2 + V(z) \right] \psi &= -\mathcal{Q}\psi, \\ V &= \frac{1}{4} \partial_z^2 \log f + \frac{1}{16} (\partial_z \log f)^2. \end{aligned} \quad (6.7)$$

In order to determine the qualitative behavior of the spectrum, it now suffices to analyze the “Schrödinger potential” $V(z)$ near the endpoints.

The coordinate z was chosen as an increasing function of y . The potential has the following properties. Firstly, at the AdS boundary, which is reached as $y \rightarrow \infty$, we have $z \rightarrow z'$, where z' is finite, and the potential blows up quadratically:

$$\text{AdS boundary : } V = \frac{C_N}{(z - z')^2}. \quad (6.8)$$

Here, the coefficient C_N depends on N (but not on n_e). At the “horizon” boundary $y \rightarrow 0$, we have $z \rightarrow \tilde{z}$ with \tilde{z} finite for $n_e = 1, 2, 3$, whilst $\tilde{z} \rightarrow -\infty$ for $n_e = 4, 5, 6, 7$. At this boundary the potential has the form

$$\begin{aligned} \text{Horizon boundary : } \quad V &= \frac{C_{n_e}}{(z - \tilde{z})^2}, \quad n_e = 1, 2, 3, \\ V &\rightarrow 1, \quad n_e = 4, \\ V &\rightarrow 0, \quad n_e = 5, 6, 7, \end{aligned} \quad (6.9)$$

where the coefficient is C_{n_e} is independent of N and depends on n_e in the following way:

$$C_{n_e} = \frac{n_e(3n_e - 8)}{4(n_e - 4)^2}. \quad (6.10)$$

The value of $C_{n_e} = \{-\frac{5}{36}, -\frac{1}{4}, \frac{3}{4}\}$ for $n_e = \{1, 2, 3\}$ respectively.

Thus barring the subtlety that the potential should not have any additional local minimum, the spectra have the following universal features. For the cases $n_e = 1, 2, 3$, the potentials give rise to discrete spectra. When $n_e = 4$, the potential gives rise to a continuous spectrum above a gap, whilst for $n_e = 5, 6, 7$ the spectra are continuous without any gap. (Note that had C_{n_e} been less than $-\frac{1}{4}$, the spectrum would have been unbounded from below. The case $n_e = 2$ with $C_2 = -\frac{1}{4}$ is precisely on the borderline.)

Numerical plots of the various potentials are displayed in Figures 1-7, for $n_e = 1, \dots, 7$, respectively. In order to facilitate the comparison of the potentials for a given n_e but different values of N , in Figures 1-3, the additive ambiguity in the definition of z has been fixed by requiring that $\tilde{z}_i = 0$ for all N , and thus the “horizon” boundary corresponds to

$z = 0$ for all N . The z coordinate was subsequently rescaled so that $z = 1$ corresponds to the boundary of AdS for all values of N . In Figures 4-7, the additive ambiguity in z has been fixed such that $z_N = 0$, and thus $z = 0$ corresponds to the AdS boundary for all N , while $z \rightarrow -\infty$ is the horizon boundary. The numerically evaluated potentials, displayed in Figures 1-7, have no additional local minima, thus confirming the nature of the spectra described above. For each value of n_e the universal behavior at the horizon boundary is apparent, and the potentials increase monotonically with decreasing N . Thus for each $n_e = 1, 2, 3$ the discrete spectra have an increasing gap between the discrete values of \mathcal{Q} for the M2-brane (solid lines), D3-brane (dotted lines), D4-D8-brane (dash lines) and M5-brane (dash-dotted lines) respectively. The rather universal patterns for the spectra for each n_e is also reflected in the patterns for the charge distributions given in the previous section. Note also that while the geometries with $n_e \leq 4$ have “naked” singularities, nevertheless the spectra are regular (and are *not unbounded* from below).

A few additional comments are in order: The D3-brane case ($N = 6$) was analyzed in [7].⁷ The integer n used in [7] is the same as our n for the Lorentz parameters, and thus our n_e is related to the n in [7] by $n_e = 6 - n$. We found no unbounded spectrum for $n_e = 1$.

6.2 Spectrum with “Lorentzian” parameters

Since the case with n equal nonzero Lorentzian parameters $\ell_1^2 = \ell_2^2 = \dots = \ell_n^2 > 0$ is equivalent to the case with $n_e = N - n$ Euclidean nonzero parameters, for the M2-brane ($N = 8$), D3-brane ($N = 6$) and M5-brane ($N = 5$),⁸ all these cases have already been encompassed in the study above.

On the other hand the D4-D8-brane does not have this symmetry (note that in $D = 6$, we have from (2.19) that $N = \frac{16}{3}$, which is a non-integer) and the analysis for Lorentzian parameters has to be done separately for these examples. The wave equation (6.3) in this case can be cast in the form (6.4) where $y = r^2$ runs from 0 (at the horizon) to ∞ (the boundary of AdS) with the function $f(y)$ taking the form:

$$f(y) = 4y^{\frac{1}{2}(N-n)} (y+1)^{\frac{1}{2}n}, \quad (6.11)$$

⁷Note also in this case the example with $n_e = 4$ corresponds to the study of glue-ball (0^{++}) spectra in Euclidean spinning-brane backgrounds with two angular momenta, which were studied in [17]. There, using the WKB approximation, the first few discrete eigenvalues were found. However, the present analysis shows that the spectrum becomes continuous for higher excitations.

⁸This is easily seen by performing a coordinate transformation on either a solution (or the wave equation (6.2)): $r^2 \rightarrow r^2 + \ell_i^2$. Thus both the distributions of the brane configurations (see previous section) and the spectra in these backgrounds are identical.

with $N = \frac{16}{3}$. The quantity f is again an increasing positive function of y , with the asymptotic behavior

$$\begin{aligned} f(y) &\sim y^{\frac{1}{2}N} && \text{for } y \rightarrow \infty, \\ f(y) &\sim y^{\frac{1}{2}(N-n)} && \text{for } y \rightarrow 0. \end{aligned} \quad (6.12)$$

Here $n = 1, \dots, 4$ is the number of (equal) non-zero parameters $\ell_i^2 \geq 0$ whose values we took (without loss of generality) to be $+1$, *i.e.* $\ell_1^2 = \dots = \ell_n^2 = 1$. (In the case of $\ell_i \neq \ell_j$ the analysis of the spectra is qualitatively the same as for the examples with $\max(\ell_i^2) = 1$, and the other parameters set to zero.)

Using the procedure, outlined above, to obtain the ‘‘Schrödinger potential’’ $V(z)$, one arrives at the following features of the potential $V(z)$ at the endpoints. The AdS boundary is again reached as $y \rightarrow \infty$. At this boundary we have $z \rightarrow z'$, with z' finite, and the potential blows up quadratically as in (6.8), with coefficient C_N independent of n . At the horizon boundary ($y \rightarrow 0$), we have $z \rightarrow \tilde{z}$ with \tilde{z} finite for $n = 2, 3, 4$ and $\tilde{z} = -\infty$ for $n = 1$. At this boundary the potential has the form

$$\begin{aligned} \text{Horizon boundary :} \quad V &\rightarrow 0 \quad n = 1, \\ V &= \frac{C_{\tilde{n}}}{(z - \tilde{z})^2}, \quad n = 2, 3, 4. \end{aligned} \quad (6.13)$$

where $\tilde{n} = N - n$ and the coefficient is $C_{\tilde{n}}$ takes the same form as (6.10) but with n_e replaced by \tilde{n} . Thus for the D4-D8-brane with $N = \frac{16}{3}$, we find that when $n = 1$ there is a continuous spectrum without a gap, while the spectra for $n = 2, 3, 4$ are discrete, with the corresponding values $C_{\tilde{n}} = [+ \frac{15}{4}, - \frac{21}{100}, - \frac{3}{16}]$ for the coefficients. (Note again that since $C_{\tilde{n}} > -\frac{1}{4}$, the spectrum is always bounded from below.)

The numerically evaluated potentials are displayed in Figures 1-4, for $n = 4, 3, 2, 1$, respectively, as another (bold) solid line. These potentials are quantitatively different in behavior from the respective potentials with $n_e = 1, 2, 3, 4$. Nevertheless, it appears that the cases with n and $n_e = 5 - n$ are qualitatively similar.

Note Added

When this paper was nearing completion, the work [36] appeared, which has some overlap with our results in the case of D3-branes.

Acknowledgements

We thank D. Freedman, S. Giddings, K. Pilch, and N. Warner for useful discussions. The research of M.C. was in part by DOE grant DOE-FG02-95ER40893, and also in part by the University of Pennsylvania Research Foundation. The research of S.S.G. was supported by the Harvard Society of Fellows, and also in part by the NSF under grant number PHY-98-02709, and by DOE grant DE-FGO2-91ER40654. The research of H.L. was supported in part by DOE grant DOE-FG02-95ER40893. The research of C.N.P. was supported by DOE grant DOE-FG03-95ER40917. S.S.G. also thanks the Aspen Center for Physics for hospitality. The authors thank the ICTP and SISSA for hospitality during the initial phases of the work.

References

- [1] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.
- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** (1998) 105, hep-th/9802109.
- [3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.
- [4] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics*, JHEP **12** (1998) 022, hep-th/9810126.
- [5] J. Distler and F. Zamora, *Non-supersymmetric conformal field theories from stable anti-de Sitter spaces* Adv. Theor. Math. Phys. **2**, (1999) 1405, hep-th/9810206.
- [6] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, hep-th/9904017.
- [7] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Continuous distributions of D3-branes and gauged supergravity*, hep-th/9906194.
- [8] K. Behrndt, *Domain walls of $D = 5$ supergravity and fixpoints of $N = 1$ super Yang Mills*, hep-th/9907070.

- [9] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, hep-th/9905104.
- [10] H. Nastase, D. Vaman and P. van Nieuwenhuizen, *Consistent nonlinear KK reduction of 11d supergravity on $AdS_7 \times S_4$ and self-duality in odd dimensions*, hep-th/9905075.
- [11] H. Lü, C.N. Pope, *Exact embedding of $N = 1$, $D = 7$ gauged supergravity in $D = 11$* , to appear in Phys. Lett. **B**, hep-th/9906168.
- [12] M. Cvetič, H. Lü and C.N. Pope, *Gauged six-dimensional supergravity from massive type IIA*, hep-th/9906221.
- [13] B. de Wit and H. Nicolai, *$N = 8$ supergravity with local $SO(8) \times SU(8)$ invariance*, Phys. Lett. **B108** (1982) 285.
- [14] B. de Wit and H. Nicolai, *$N=8$ supergravity*, Nucl. Phys. **B208** (1982) 323.
- [15] M. Cvetič, M.J. Duff, P. Hoxha, J.T. Liu, H. Lü, J.X. Lu, R. Martinez-Acosta, C.N. Pope, H. Sati and T.A. Tran, *Embedding AdS black holes in ten and eleven dimensions*, hep-th/9903214, to appear in Nucl. Phys. **B**.
- [16] P. Kraus, F. Larsen and S.P. Trivedi, *The Coulomb branch of gauge theory from rotating branes*, JHEP **9903** (1999) 003, hep-th/9811120.
- [17] J.G. Russo and K. Sfetsos, *Rotating D3-branes and QCD in three dimensions*, Adv. Theor. Math. Phys. **3** (1999) 131, hep-th/9901056.
- [18] M.J. Duff and J. Liu, *Anti-de Sitter black holes in gauged $N = 8$ supergravity*, Nucl. Phys. **B554** (1999) 237, hep-th/9901149.
- [19] M. Cvetič and H.H. Soleng, *Supergravity domain walls*, Phys. Rep. **282** (1997) 159, hep-th/9604090.
- [20] K. Behrndt, M. Cvetič, W.A. Sabra, *Non-extreme black holes of five-dimensional $N = 2$ AdS supergravity*, Nucl. Phys. **B553** (1999) 317.
- [21] A. Brandhuber and K. Sfetsos, *Wilson loops from multicentre and rotating branes, mass gaps and phase structure in gauge theories*, hep-th/9906201.
- [22] I. Chepelev and R. Roiban, *A note on correlation functions in $AdS(5)/SYM(4)$ correspondence on the Coulomb branch*, hep-th/9906224.

- [23] S.B. Giddings and S.F. Ross, *D3-brane shells to black branes on the Coulomb branch*, hep-th/9907204.
- [24] L.J. Romans, *The $F(4)$ gauged supergravity in six dimensions*, Nucl. Phys. **B269** (1986) 691.
- [25] S. Ferrara, A. Kehagias, H. Partouche and A. Zaffaroni, *AdS₆ interpretation of 5-D superconformal field theories*, Phys. Lett. **B431** (1998) 57, hep-th/9804006.
- [26] L.J. Romans, *Massive $N = 2a$ supergravity in ten dimensions*, Phys. Lett. **B169** (1986) 374.
- [27] H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, *Dilatonic p -brane solutions*, Phys. Lett. **B371** (1996) 46, hep-th/9511203.
- [28] K. Skenderis and P.K. Townsend, *Gravitational stability and renormalization group flow*, hep-th/9909070.
- [29] D. Youm, *Localised intersecting BPS branes*, hep-th/9902208
- [30] A. Brandhuber and Y. Oz, *the D4-D8 brane system and five-dimensional fixed points*, Phys. Lett. **B460** (1999) 307, hep-th/9905148.
- [31] C. Csaki, Y. Oz, J. Russo and J. Terning, *Large N QCD from rotating branes*, Phys. Rev. **D59** (1999) 065012, hep-th/9810186.
- [32] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, Phys. Rev. **D60** (1999) 064018, hep-th/9902170.
- [33] M. Cvetič and S.S. Gubser, *Phases of R-charged black holes, spinning branes and strongly coupled gauge theories*, JHEP **04** (1999) 024, hep-th/9902195.
- [34] M. Cvetič and S.S. Gubser, *Thermodynamic stability and phases of general spinning branes*, JHEP **07** (1999) 010, hep-th/9903132.
- [35] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories* Adv. Theor. Math. Phys. **2** (1998) 505, hep-th/9803131.
- [36] I. Bakas and K. Sfetsos, *States and curves of five-dimensional gauged supergravity*, hep-th/9909041.

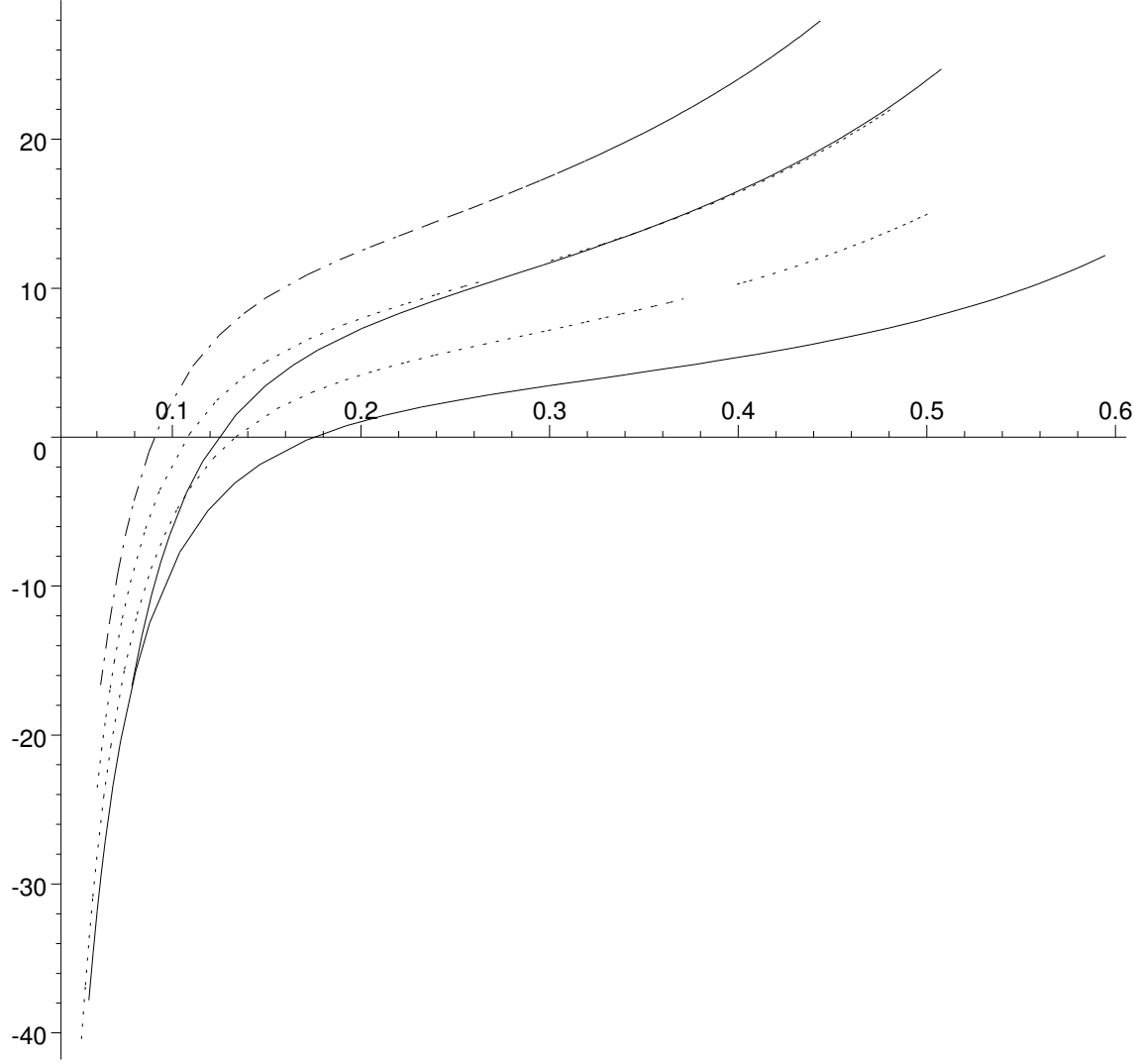


Figure 1: The Schrödinger potentials $V(z)$ with $n_e = 1$ Euclidean parameters are given for M2-branes, D3-branes, D4-D8-branes and M5-branes, with successively increasing values of the potential. The additional solid line represents the D4-D8-brane with $n = 4$ equal Lorentz parameters. The additive ambiguity for z is fixed in such a way that $z = 0$ corresponds to the horizon boundary, and z is rescaled so that $z = 1$ is the boundary of AdS in all cases.

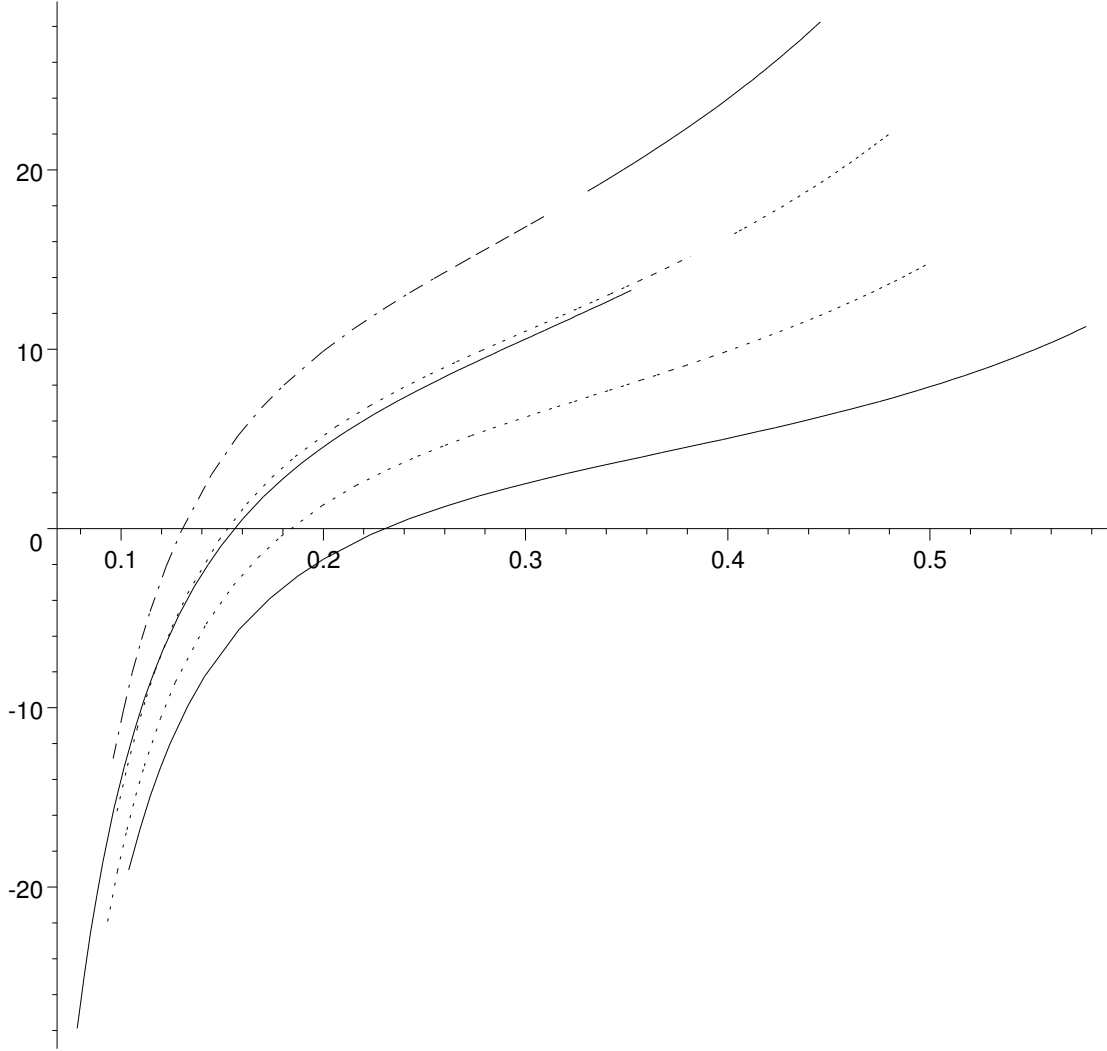


Figure 2: The Schrödinger potentials $V(z)$ with $n_e = 2$ Euclidean parameters are given for M2-branes, D3-branes, D4-D8-branes and M5-branes, with successively increasing values of the potential. The additional solid line represents the D4-D8-brane with $n = 3$ equal Lorentz parameters. The additive ambiguity for the z is fixed in such a way that $z = 0$ corresponds to the horizon boundary, and z is rescaled so that $z = 1$ is the boundary of AdS in all cases.

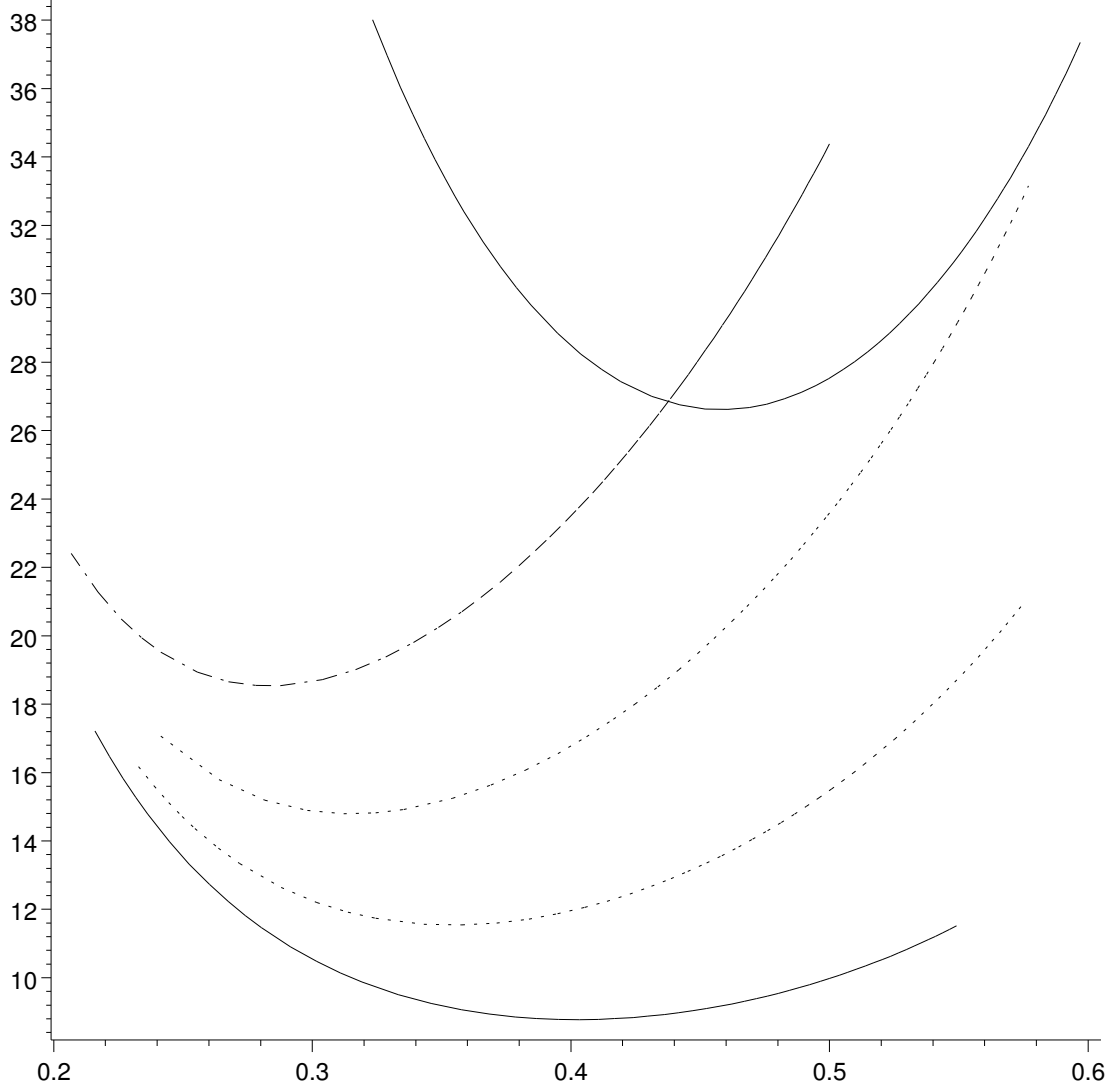


Figure 3: The Schrödinger potentials $V(z)$ with $n_e = 3$ Euclidean parameters are given for M2-branes, D3-branes, D4-D8-branes and M5-branes, with successively increasing values of the potential. The additional solid line represents the D4-D8-brane with $n = 2$ equal Lorentz parameters. The additive ambiguity for z is fixed in such a way that $z = 0$ corresponds to the horizon boundary, and z is rescaled so that $z = 1$ is the boundary of AdS in all cases.

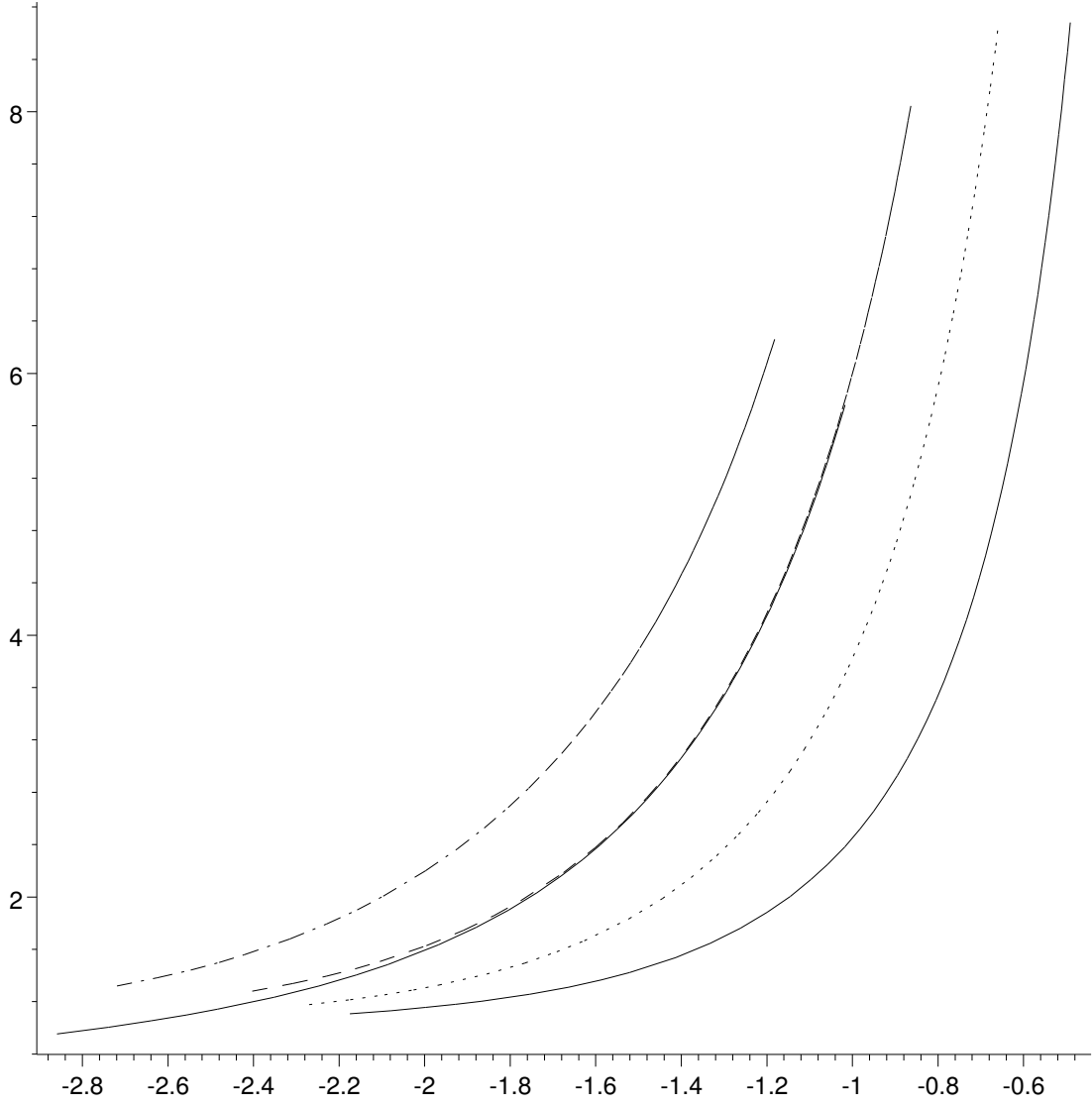


Figure 4: The Schrödinger potentials $V(z)$ with $n_e = 4$ Euclidean parameters are given for M2-branes, D3-branes, D4-D8-branes and M5-branes, with successively increasing values of the potential. The additional solid line represents the D4-D8-brane with $n = 1$ Lorentz parameter. $z = 0$ corresponds to the AdS boundary. $V \rightarrow 1$ for the $n_e = 4$ cases and $V \rightarrow 0$ for $n = 1$.

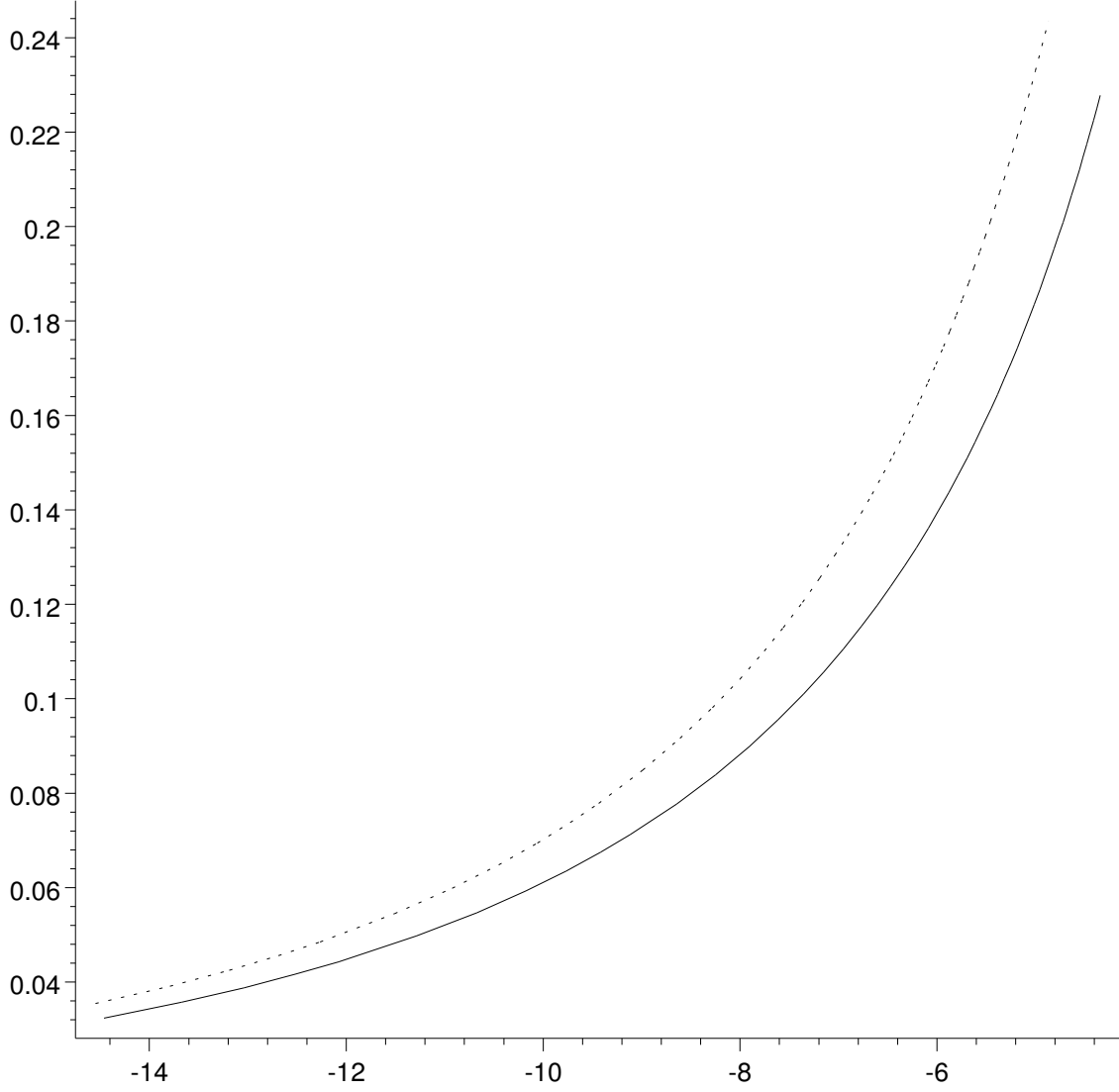


Figure 5: The Schrödinger potentials $V(z)$ with $n_e = 5$ Euclidean parameters are given for M2-branes and D3-branes, with successively increasing values of the potential. The AdS boundary is at $z = 0$.

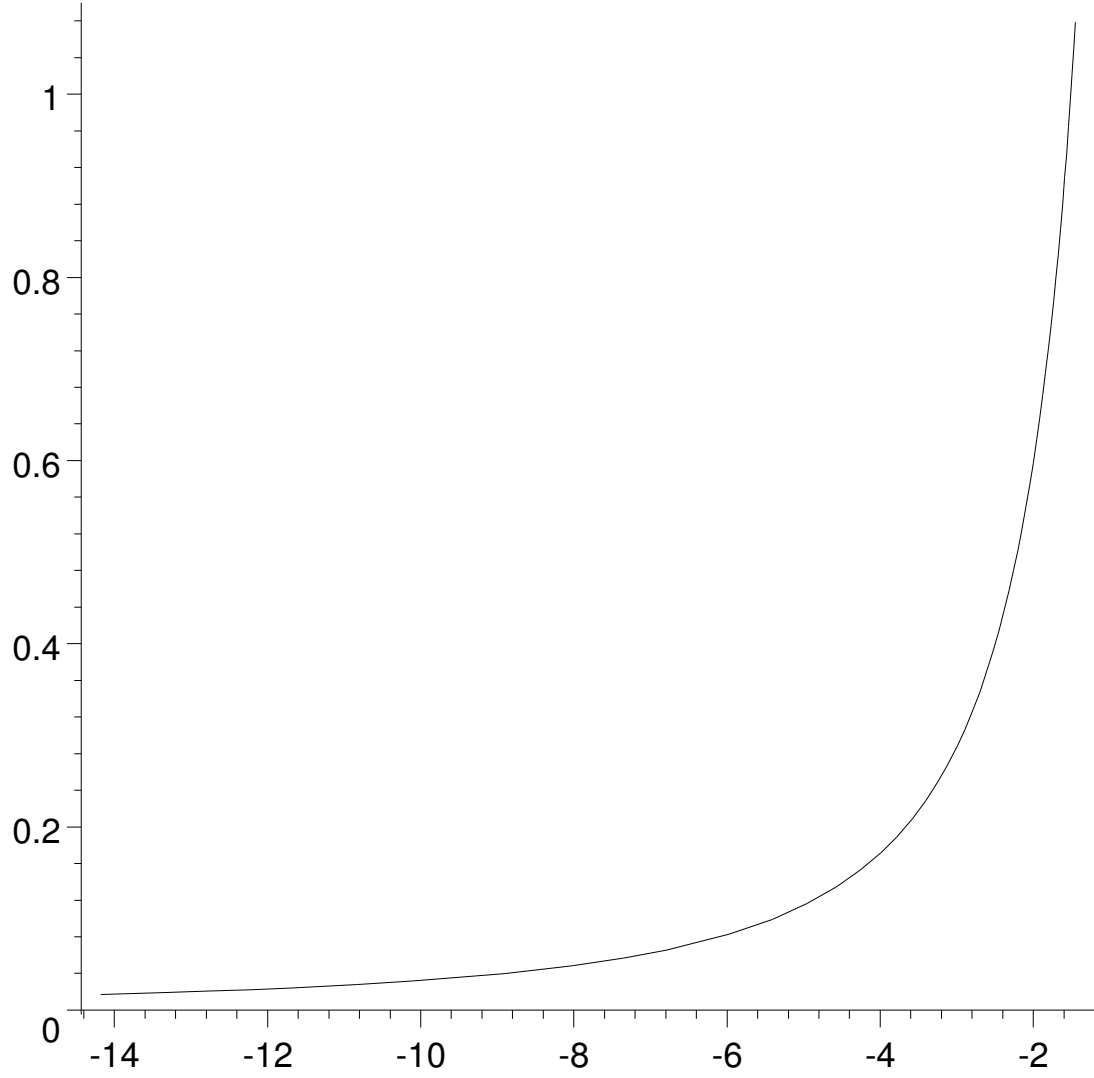


Figure 6: The Schrödinger potential $V(z)$ with $n_e = 6$ Euclidean parameters is given for the M2-brane. The AdS boundary is at $z = 0$.

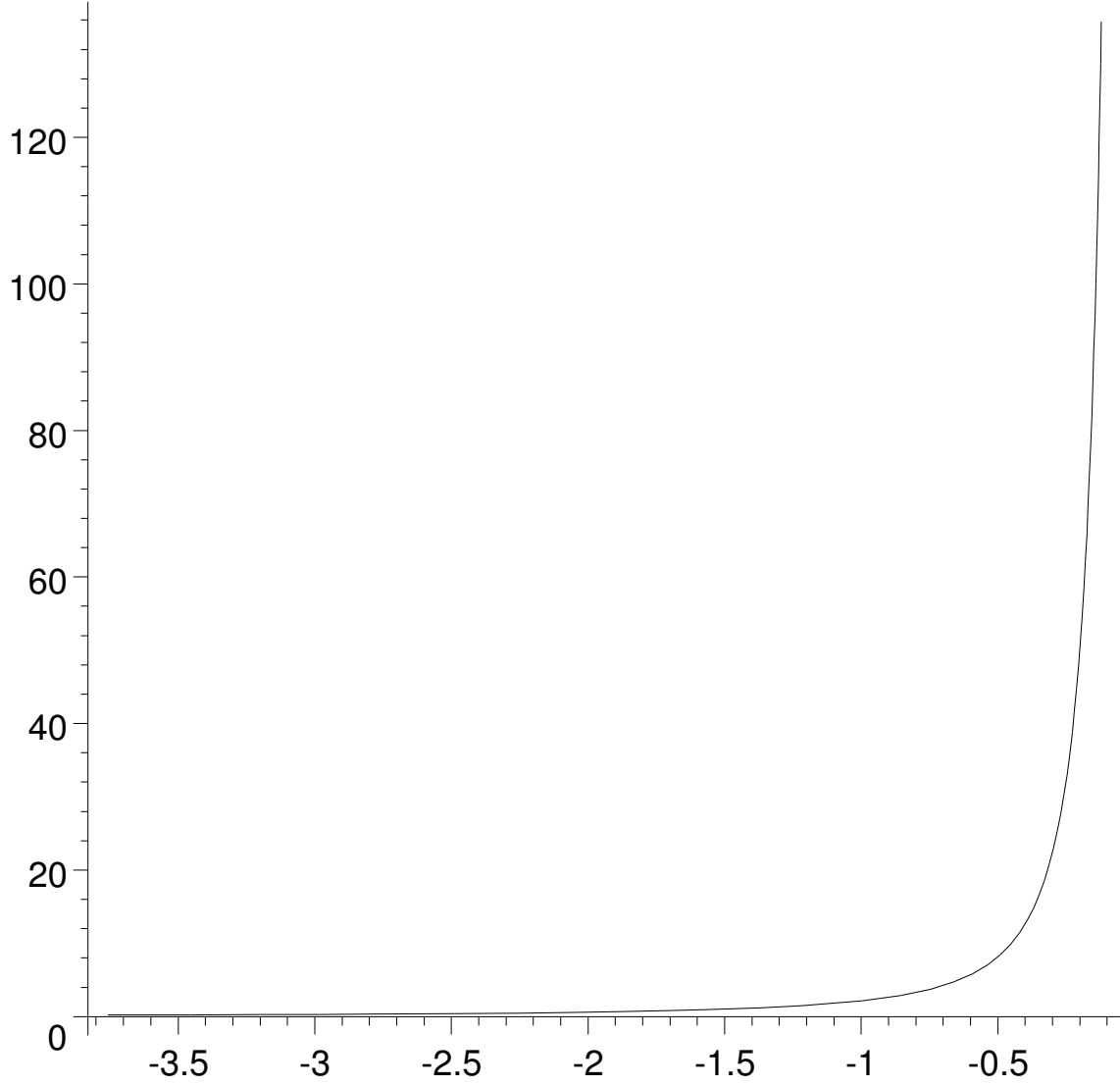


Figure 7: The Schrödinger potential $V(z)$ with $n_e = 7$ Euclidean parameters is given for the M2-brane. The AdS boundary is at $z = 0$.